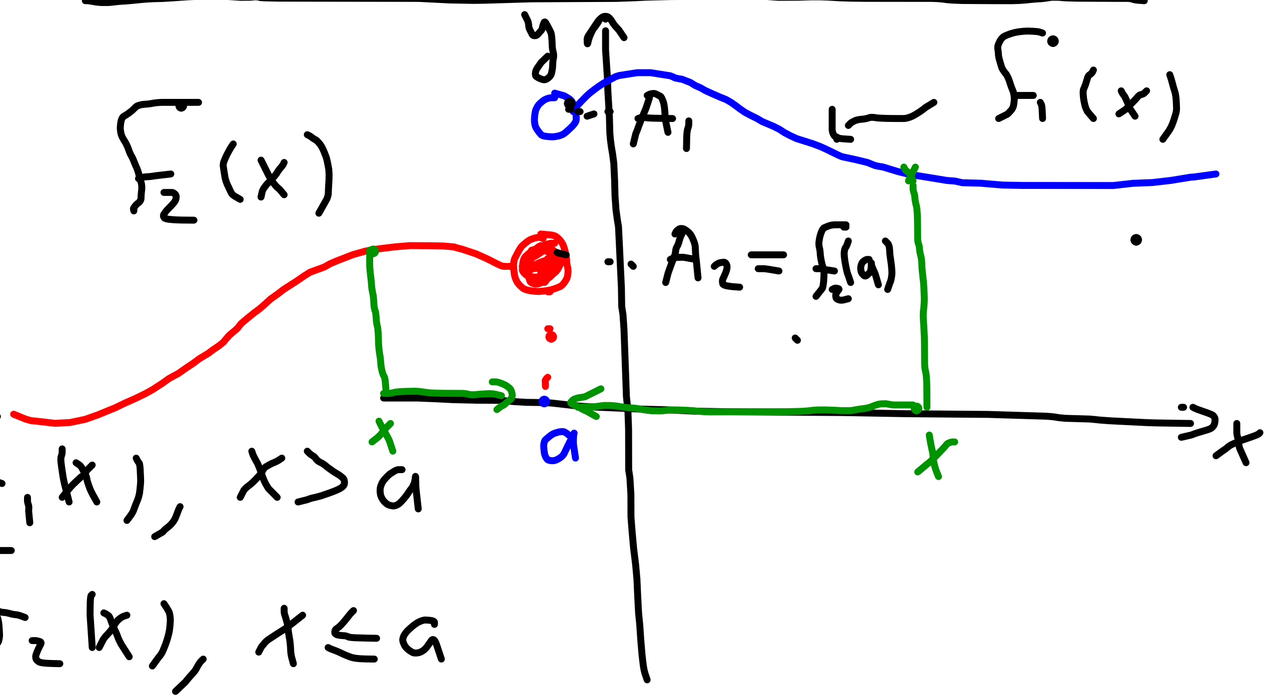
Förel. nr 4 GRÄNSVÄRDEN AV funktioner

Def 1.



$$f(x) = \begin{cases} f_1(x), & x > a \\ f_2(x), & x \le a \end{cases}$$

a) f säges ha gränsvärdet A_1 från höger till punkten $x = a$ om $\lim_{x \to a+} f(x) = A_1$

b) Vi säger att f har gränsvärdet A_2 från vänstra till $x=a$

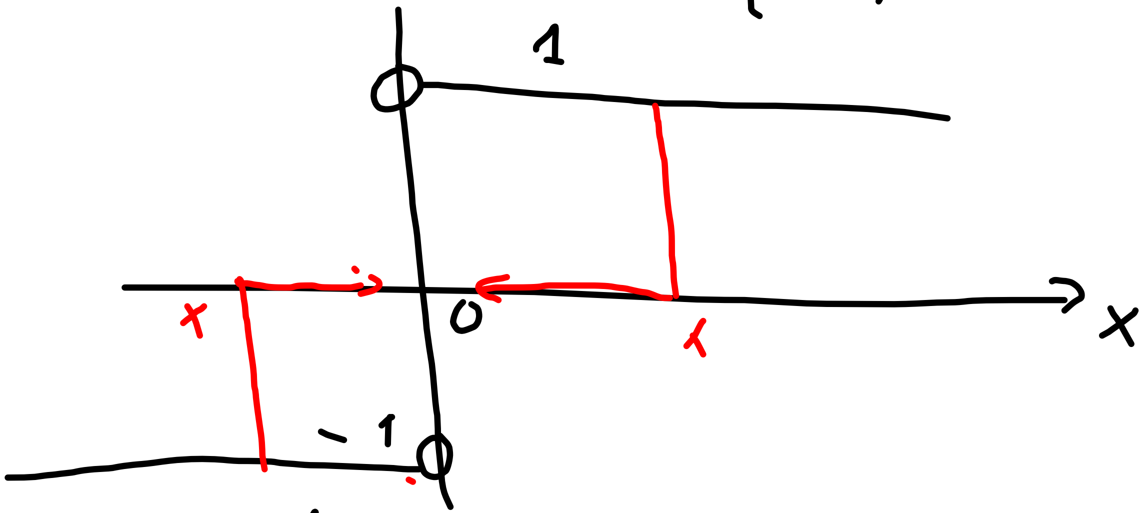
om $\lim_{x \to a-} f(x) = A_2$

c) f säges ha gränsvärdet A i $x=a$ om

$$\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = A$$

Ex 1

$$f(x) = \operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

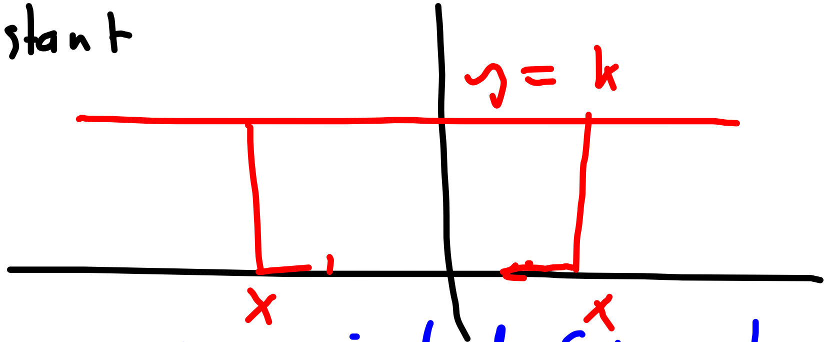


f ej definierad i $x = 0$

$$\lim_{x \to 0+} f(x) = 1 \neq \lim_{x \to 0-} f(x) = -1$$

$\lim_{x \to 0} f(x)$ saknas.

Ex2 $f(x) = k = \text{konstant}$



$$\lim_{x\to a} f(x) = k$$

Här gränsvärdet finns!

Ex3 $-1 \le \sin x \le 1$

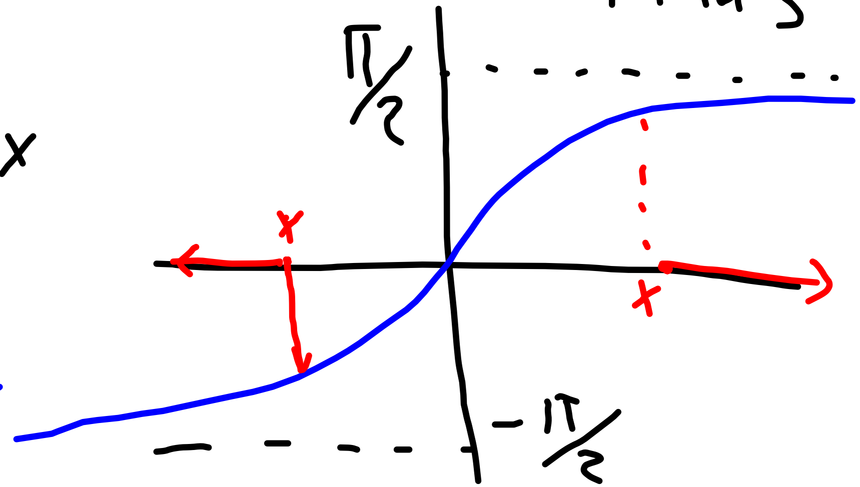
$$\lim_{x\to\pm\infty} \sin x = \begin{cases} +1 \\ -1 \end{cases}$$

$\therefore$ gränsvärdet saknas

Ex4 $f(x) = \arctan x$

$$\lim_{x\to\infty} \arctan x = \pi/2$$

$$\lim_{x\to-\infty} \arctan x = -\pi/2$$



Ex 5 $\lim_{x\to\infty} \frac{x^2+x+1}{x^2+2x+2}$

Lösning $x>0$, Förenkla

$$\frac{x^2+x+1}{x^2+2x+2} = \frac{\cancel{x^2}\left(1+\frac{\cancel{x}}{x^2}+\frac{1}{x^2}\right)}{\cancel{x^2}\left(1+\frac{2\cancel{x}}{\cancel{x^2}}+\frac{2}{x^2}\right)}$$

$\lim_{x\to\infty} \frac{1}{x} = 0$, $\lim_{x\to\infty} \frac{1}{x^2} = 0$

$$\lim_{x\to\infty} \frac{x^2+x+1}{x^2+2x+2} = \lim_{x\to\infty} \frac{1+\frac{1}{x}+\frac{1}{x^2}}{1+\frac{2}{x}+\frac{2}{x^2}} = \frac{1+0+0}{1+0+0} = 1$$

Lai in EX $\lim_{x\to\infty} \sqrt{x^2+x} - x$

Lösning "Vanligt fel"

$$\underbrace{\sqrt{x^2+x}}_{\to\infty} - \underbrace{x}_{\to\infty} = \underbrace{\infty - \infty}_{\text{odef.}} = 0 \text{ Fel!}$$

① FÖRENKLA via konjugatregel

$$(a-b)(a+b) = a^2-b^2$$

$x>0$

$$\underbrace{\sqrt{x^2+x}}_{a} - \underbrace{x}_{b} = \frac{\left(\sqrt{x^2+x}-x\right)\left(\sqrt{x^2+x}+x\right)}{\sqrt{x^2+x}+x}$$

$$= \frac{\left(\sqrt{x^2+x}\right)^2 - x^2}{\sqrt{x^2+x}+x} = \frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x} = \frac{x}{\sqrt{x^2+x}+x}$$

$$= \frac{x}{\sqrt{x^2+x} + x} = \left[\sqrt{\frac{x^2+x}{\sqrt{x^2}\sqrt{1+\frac{1}{x}}}} = \sqrt{x^2\left(1+\frac{1}{x}\right)} = \right]$$

$$= \frac{x}{\sqrt{x^2}\sqrt{1+\frac{1}{x}} + x} = \left[\sqrt{x^2} = \begin{cases} x > 0 \\ -x, \; x < 0 \end{cases} \right]$$

$$= \left[\text{i vårt fall, } x > 0 \right] = \frac{\cancel{x}}{\cancel{x}\sqrt{1+\frac{1}{x}} + \cancel{x}}$$

$$= \frac{1}{\sqrt{1+\frac{1}{x}} + 1}$$

$$\therefore \lim_{x \to \infty} \left(\sqrt{x^2+x} - x \right) = \lim_{x \to \infty} \frac{1}{\sqrt{1+\frac{1}{x}} + 1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$$

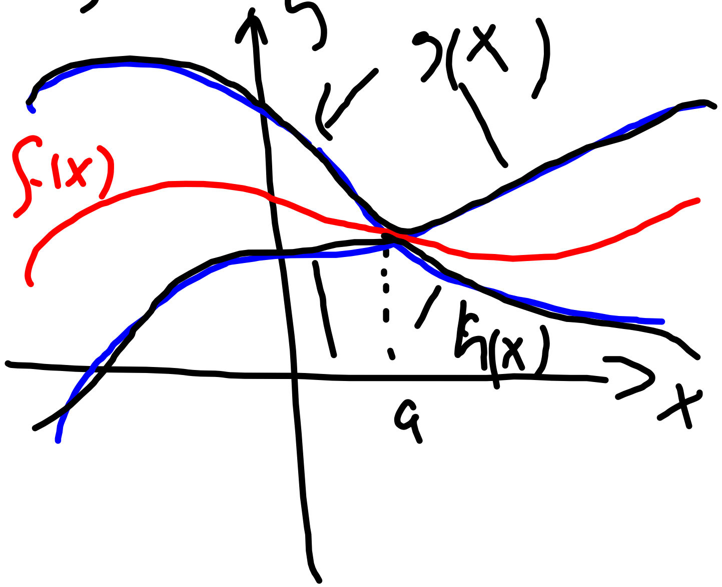
<u>Räknelagar</u> finns på sid 136

<u>Speciellt</u> <u>Instängningsregeln</u>

(1) $h(x) \le f(x) \le g(x)$

(2) $\lim_{x \to a} h(x) = \lim_{x \to a} g(x) = A$

$\Rightarrow \lim_{x \to a} f(x) = A$



Ex $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$

Lösning $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = \left[\begin{array}{l} t = \frac{1}{x} \\ t \to \infty \Longleftrightarrow x \to 0 \end{array} \right]$

$= \lim_{t \to \infty} \frac{1}{t^2} \sin(t)$

[Instängningsregel] $-1 \le \sin t \le 1$

$$\underbrace{\frac{-1}{t^2}}_{\substack{\downarrow \\ 0}} \le \frac{\sin t}{t^2} \le \underbrace{\frac{1}{t^2}}_{\substack{\downarrow \\ 0}} \Rightarrow \lim_{t \to \infty} \frac{\sin t}{t^2} = 0$$

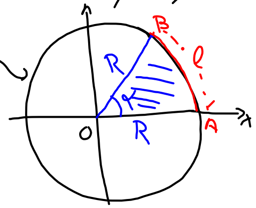
$t \to \infty$ $\quad$ $t \to \infty$

Viktig sats $\lim_{x\to 0}\frac{\sin x}{x}=1$

Som används för att TA fram derivatorna till $\sin x$, $\cos x$, osv.

Motivering

$x^2+y^2=R^2$

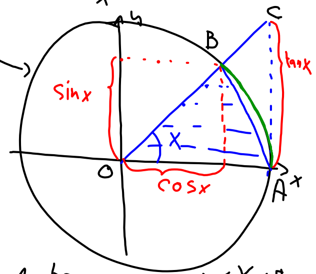


ℓ = längden av bågen AB $= R\alpha$

Arean av sektorn $OAB = \frac{\alpha}{2}R^2$

Vi motiverar $\lim_{x\to 0}\frac{\sin x}{x}=1$

$x^2+y^2=1$



arean av triangel OBA < arean av sektorn OBA < arean av triangel OCA dvs

$$\frac{1\cdot\sin x}{2} < \frac{x}{2}1^2 < \frac{1\cdot\tan x}{2} \qquad 0<x<\pi/2$$

$$\Longleftrightarrow \boxed{\sin x < x < \tan x} \quad (*)$$

ur $(*)$ $\sin x < x \Longleftrightarrow \frac{\sin x}{x} < 1$ (1)

ur $(*)$ $x < \tan x \Longleftrightarrow x < \frac{\sin x}{\cos x} \Longleftrightarrow \cos x < \frac{\sin x}{x}$ (2)

ur (1) och (2)

$$\cos x < \frac{\sin x}{x} < 1 \qquad 0<x<\pi/2$$

Vidare

$$\underbrace{\cos(-x)}_{\cos x} < \frac{\sin(-x)}{-x} < 1 \qquad -\pi/2<x<0$$

$$< \frac{\sin x}{x} < 1$$

$\therefore$

$$\cos x < \frac{\sin x}{x} < 1 \qquad -\pi/2<x<\pi/2,\ x\neq 0$$

$$\underbrace{\lim_{x\to 0}\cos x}_{=1} < \lim_{x\to 0}\frac{\sin x}{x} < \underbrace{\lim_{x\to 0} 1}_{=1}$$

Instängningsregeln $\Longrightarrow \lim_{x\to 0}\frac{\sin x}{x}=1$

Tillämpning

$a \neq 0$, $\displaystyle\lim_{x\to a} \frac{\sin(x-a)}{x^2-a^2} = \left[\text{tänk } \lim_{t\to 0} \frac{\sin t}{t} = 1\right]$

$$\lim_{x\to a} \frac{\sin(x-a)}{(x-a)(x+a)} = \left[\begin{matrix} x-a=t \\ x\to a \iff t\to 0\end{matrix}\right]$$

$$= \lim_{t\to 0} \frac{\sin t}{t} \cdot \frac{1}{t+2a} = \underbrace{\left(\lim_{t\to 0}\frac{\sin t}{t}\right)}_{=1} \underbrace{\left(\lim_{t\to 0}\frac{1}{t+2a}\right)}_{\frac{1}{0+2a} = 1/2a}$$

Svar $\displaystyle\lim_{x\to a} \frac{\sin(x-a)}{x^2-a^2} = \frac{1}{2a}$, $a \neq 0$

Ex $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2} = \left[\text{inte sant att skriva } \frac{1-\cos(0)}{0^2} = \frac{0}{0} \text{ odef}\right]$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \left[(a-b)(a+b) = a^2-b^2\right]$$

$a = 1$, $b = \cos x$

$$= \lim_{x\to 0} \frac{(1-\cos x)}{x^2} \frac{(1+\cos x)}{1+\cos x} = \lim_{x\to 0} \left(\frac{\overbrace{1-\cos^2 x}^{\sin^2 x}}{x^2}\right)\left(\frac{1}{1+\cos x}\right)$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x^2} \frac{1}{1+\cos x} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \frac{1}{1+\cos x}$$

$$= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 \lim_{x\to 0} \frac{1}{1+\cos x}$$

$$= 1^2 \cdot \frac{1}{1+\cos 0} = 1 \cdot \frac{1}{1+1} = 1/2$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = 1/2$$

Tillämpning av gränsvärden: kontinuitet

Def Vi säger att funktionen f är kontinuerlig i $x = a$

om ① $\lim_{x \to a} f(x) = A$

② $f(a) = A$

① och ② f kontinuerlig i $x = a$

$\Longleftrightarrow \lim_{x \to a} f(x) = f(a)$

Typiska problem

Problem 1

$$f(x) = \begin{cases} g(x), & x \neq a \\ A, & x = a \end{cases}$$

f blir kontinuerlig i $x = a$ om

$$\lim_{x \to a} g(x) = A = f(a)$$

Problem 2

$$f(x) = \begin{cases} f_1(x), & x > a \\ f_2(x), & x < a \end{cases}$$

f blir kontinuerlig i $x = a$

om $\lim_{x \to a+} f_1(x) = \lim_{x \to a-} f_2(x) = f(a)$

Alla elementära f-ner är kontinuerliga

$\sin x$, $\cos x$, $\tan x$, där $\cos x \neq 0$

$\cot x$, där $\sin x \neq 0$, $\ln x$, e^x, polynom