Förel 21 1. Jämförelse mellan positiva serier och integraler

2. L'HÔPITALS REGEL

Def Serien $\sum_{k=1}^{\infty} a_k$ positiv om $a_k \geq 0$, $k=1,2,3,\dots$

Integral TEST om $f(x) \geq 0$, $x \geq a$ och $f(x)$ är avtagande (t.ex. om $f'(x) \leq 0$)

Då gäller

$$\sum_{k=1}^{\infty} f(k) \quad \text{och} \quad \int_1^{\infty} f(x)\,dx$$

konvergerar (divergerar) samtidigt

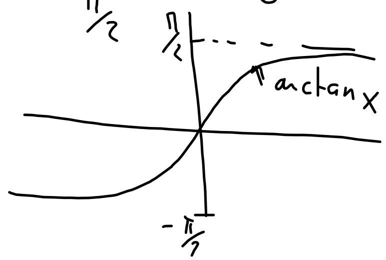
$$\sum_{k=1}^{\infty} f(k) \text{ konvergerar} \iff \int_1^{\infty} f(x)\,dx \text{ konvergerar}$$

EX1

$$\sum_{k=0}^{\infty} \frac{1}{k^2+1} \sim \int_0^{\infty} \frac{1}{x^2+1}\,dx =$$

$$= \lim_{b\to\infty} \int_0^b \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \left[\arctan x\right]_0^b$$

$$= \underbrace{\lim_{b\to\infty} \arctan(b)}_{\pi/2} - \underbrace{\arctan(0)}_{=0}$$



∴ Enligt IntegralTest är Serien $\sum_{k=0}^{\infty} \frac{1}{k^2+1}$ konvergent

EX 2 $\sum_{k=1}^{\infty} \frac{1}{k} \sim \int_1^{\infty} \frac{1}{x} dx =$

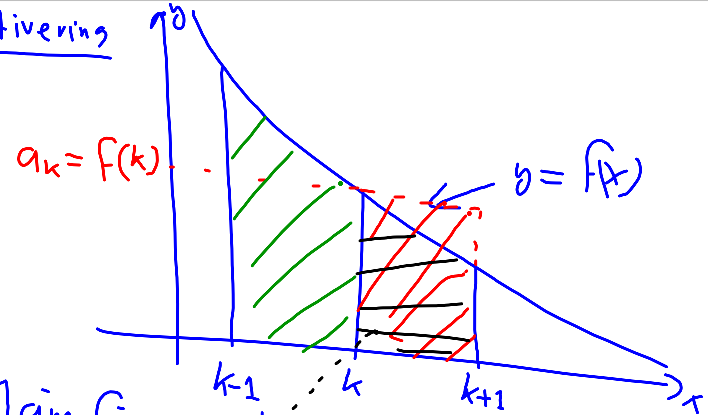
$$= \lim_{b \to \infty} \int_1^b \frac{1}{x} dx = \lim_{b \to \infty} \left[\ln x \right]_1^b = \lim_{b \to \infty} \ln(b) = \infty$$

EX3 (ATT kunna)

$$\sum_{k=1}^{\infty} \frac{1}{k^\alpha} \sim \int_1^{\infty} \frac{1}{x^\alpha} dx = \begin{cases} \frac{1}{\alpha - 1}, & \alpha > 1 \\ \infty, & \alpha \leq 1 \end{cases}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^\alpha} \begin{cases} \text{konv om } \alpha > 1 \\ \text{Div f.ö.} \end{cases}$$

Motivering



Jäm för ytorna

$$\int_k^{k+1} f(x)\,dx \le f(k)\underbrace{(k+1-k)}_{=1} \le \int_{k-1}^{k} f(x)\,dx$$

summera från $k=2$ till $n > k$

$$\underbrace{\sum_{k=2}^{n} \int_k^{k+1} f(x)\,dx} \le \sum_{k=2}^{n} f(k) \le \sum_{k=2}^{n} \int_{k-1}^{k} f(x)\,dx$$

$$\int_2^3 f(x)\,dx + \int_3^4 f(x)\,dx + \dots + \int_{n}^{n+1} f(x)\,dx = \int_2^{n+1} f(x)\,dx$$

$$f(1) + \int_2^{n+1} f(x)\,dx \le f(1) + \sum_{k=2}^{n} f(k) \le f(1) + \int_1^{n} f(x)\,dx$$

$$\boxed{f(1) + \int_2^{n+1} f(x)\,dx \le \sum_{k=1}^{n} f(k) \le f(1) + \int_1^{n} f(x)\,dx}$$

Hur man använder serien i praktiken

Finn den minsta antalet termer i serien $\sum_{k=1}^{\infty} \frac{1}{k^3+1}$ som behövs för att approximera summan med ett fel $< \frac{1}{2}10^{-2}$

Lösning

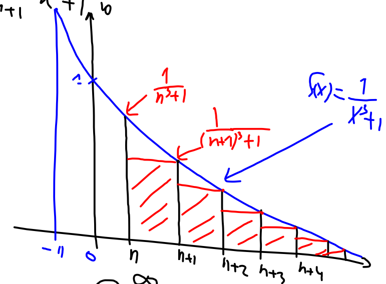
$$\sum_{k=1}^{\infty} \frac{1}{k^3+1} = \sum_{k=1}^{n} \frac{1}{k^3+1} + \underbrace{\sum_{k=n+1}^{\infty} \frac{1}{k^3+1}}_{r_n}$$

Rest termen $r_n < \frac{1}{2} 10^{-2}$

$$\sum_{k=1}^{\infty} \frac{1}{k^3+1} = \sum_{k=1}^{n} \frac{1}{k^3+1} \pm \frac{1}{2} 10^{-2}$$

Finn minsta n!

$$r_n = \sum_{k=n+1}^{\infty} \frac{1}{k^3+1}$$



$$\sum_{k=n+1}^{\infty} \frac{1}{k^3+1} \leq \int_n^{\infty} \frac{1}{x^3+1}\,dx \leq \int_n^{\infty} \frac{1}{x^3}\,dx =$$

$$= \lim_{N\to\infty} \int_n^{N} \frac{1}{x^3}\,dx = \lim_{N\to\infty} \left[-\frac{1}{2x^2}\right]_n^N$$

$$\lim_{N\to\infty} \left(-\frac{1}{2N^2} + \frac{1}{2n^2}\right) = \frac{1}{2n^2}$$

$$\therefore \sum_{k=n+1}^{\infty} \frac{1}{k^3+1} \leq \frac{1}{2n^2} < \frac{1}{2} 10^{-2} \Rightarrow n > 10$$

Svar minsta heltal $n = 11$

$$\sum_{k=1}^{\infty} \frac{1}{k^3+1} = \sum_{k=1} \frac{1}{k^3+1} \pm \frac{1}{2} 10^{-2}$$

ATT Bestämma gränsvärden med L'HÔPITALS REGEL (Ô = OS, ä, å)

Om f och g är definierad när a, $x \neq a$

och
- f, g deriverbara, $g'(a) \neq 0$
- $\lim_{x \to a} f(x) = 0$ $(\pm\infty)$ och
  $\lim_{x \to a} g(x) = 0$ $(\pm\infty)$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \neq \lim_{x \to a} \left(\frac{f(x)}{g(x)} \right)'$$

EX 1 $\lim\limits_{x\to 0} \dfrac{\arcsin x - x}{x - \arctan x}$

Lösning

$$\lim_{x\to 0} \frac{\arcsin x - x}{x - \arctan x} = \left[\frac{0}{0}\right]$$ så kan

L'HÔPITALS REGEL Användas

$$= \lim_{x\to 0} \frac{\frac{d}{dx}(\arcsin x - x)}{\frac{d}{dx}(x - \arctan x)} = \lim_{x\to 0}\left[\frac{\frac{1}{\sqrt{1-x^2}} - 1}{1 - \frac{1}{1+x^2}}\right]$$

$= \left[\frac{0}{0}\right]$ L'HÔPITALS REGEL IGEN

$$= \lim_{x\to 0} \frac{\frac{d}{dx}\left(\frac{1}{\sqrt{1-x^2}} - 1\right)}{\frac{d}{dx}\left(1 - \frac{1}{1+x^2}\right)} =$$

$$= \lim_{x\to 0} \frac{\left[\frac{d}{dx}\left[(1-x^2)^{-1/2}\right] - 0\right]}{\left[\frac{d}{dx}\left[0 - (1+x^2)^{-1}\right]\right]} = \lim_{x\to 0} \frac{\left[+\frac{1}{2}(1-x^2)^{-3/2}\,\cancel{(+2x)}\right]}{+\left(+1\,(1+x^2)^{-2}\,\cancel{(2x)}\right)}$$

$$= \lim_{x\to 0} \frac{\frac{1}{2}(1-x^2)^{-3/2}}{(1+x^2)^{-2}} = \frac{1}{2}$$

SVAR $\lim\limits_{x\to 0} \dfrac{\arcsin x - x}{x - \arctan x} = 1/2$

<u>Ex2</u> $\lim_{x\to 0} \frac{\ln(\cos x)}{x^2}$

<u>Lösning</u> $\lim_{x\to 0} \frac{\ln(\cos x)}{x^2} = \left[\frac{0}{0}\right] = \lim_{x\to 0} \frac{\frac{d}{dx}\ln(\cos x)}{\frac{d}{dx}x^2}$

$= \lim_{x\to 0} \frac{\frac{1}{\cos x}\frac{d}{dx}(\cos x)}{2x}$ $\left[\underline{\text{obs!}}\ \frac{d}{dx}\ln(u(x)) = \frac{u'(x)}{u(x)} \right]$

$= \lim_{x\to 0} \frac{\frac{-\sin x}{\cos x}}{2x} = -\lim_{x\to 0} \frac{\tan x}{2x} = \left[\frac{0}{0}\right]$

$= -\lim_{x\to 0} \frac{\frac{d}{dx}(\tan x)}{\frac{d}{dx}(2x)} = -\lim_{x\to 0} \frac{1+\tan^2 x}{2} = -1/2$

<u>SVAR</u> $\lim_{x\to 0} \frac{\ln(\cos x)}{x^2} = -1/2$

<u>EX3</u> uttryck på formen $[0 \cdot \infty]$

$$\lim_{x\to\infty}\left(\arctan x - \frac{\pi}{2}\right)x = [0\cdot\infty]$$

Omforma gränsvärdet på $\left[\frac{0}{0}\right]$

$$x = \frac{1}{\frac{1}{x}}$$

$$\lim_{x\to\infty}\left(\arctan x-\frac{\pi}{2}\right)x = \lim_{x\to\infty}\frac{\arctan x-\pi/2}{\frac{1}{x}} = \left[\frac{0}{0}\right]$$

$$= \lim_{x\to\infty}\frac{\frac{d}{dx}\left(\arctan x - \pi/2\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)} = \lim_{x\to\infty}\frac{\frac{1}{1+x^2}}{\frac{-1}{x^2}} =$$

$$= -\lim_{x\to\infty}\frac{x^2}{1+x^2} = -\lim_{x\to\infty}\frac{\cancel{x^2}}{\cancel{x^2}\left(1+\frac{1}{x^2}\right)} = -1$$

<u>SVAR</u> $\lim_{x\to\infty}\left(\arctan x-\frac{\pi}{2}\right)x = \underline{\underline{-1}}$

Uttryck på formen $[\infty - \infty]$

$$\lim_{x \to 1+} \left[\frac{x}{x-1} - \frac{1}{\ln x} \right] = [\infty - \infty]$$

Omforma på $\left[\frac{0}{0} \right]$

$$\frac{x}{x-1} - \frac{1}{\ln x} = \frac{x \ln x - 1(x-1)}{\ln x (x-1)}$$

$$\lim_{x \to 1+} \left[\frac{x}{x-1} - \frac{1}{\ln x} \right] = \lim_{x \to 1+} \left(\frac{x \ln x - (x-1)}{(x-1) \ln x} \right] = \left[\frac{0}{0} \right]$$

$$= \lim_{x \to 1+} \frac{\frac{d}{dx}\left(x \ln x - (x-1) \right)}{\frac{d}{dx}\left((x-1) \ln x \right)} = [\text{koll}] = \lim_{x \to 1+} \frac{\ln x}{\ln x + 1 - \frac{1}{x}} =$$

$$= \left(\frac{0}{0} \right] = \lim_{x \to 1+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}\left(\ln x + 1 - \frac{1}{x} \right)} =$$

$$= \lim_{x \to 1+} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{1+1} = 1/2$$

<u>SVAR</u> $\lim_{x \to 1+} \left[\frac{x}{x-1} - \frac{1}{\ln x} \right] = 1/2$