Förl. nr 2 : Logaritm - Trigonom. fkner. Binomial formeln.

1. Logaritmer med basen $a>0$ dvs ${}^a\log(x)$

Problem Få $x>0$, sök $y : a^y = x$

SVAR: $a^y = x \iff y = {}^a\log(x)$

Vi säger att $y = {}^a\log(x)$ är inversen till $y = a^x$

Obs! ${}^a\log(x)$ definierad endast då $x>0$

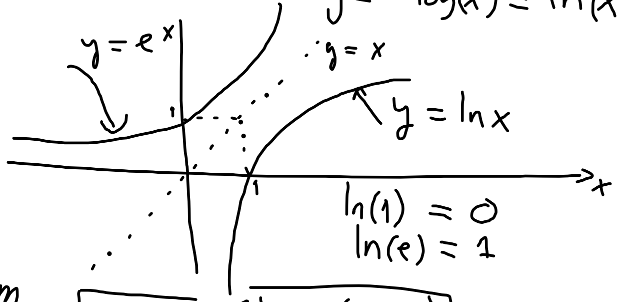
Vidare ${}^a\log(a) = 1$

Vi har

$$a^y = x \iff y = {}^a\log(x)$$

$$10^y = x \iff y = {}^{10}\log(x) = \lg(x)$$

$$e^y = x \iff y = {}^e\log(x) = \ln(x)$$



$\ln(1) = 0$

$\ln(e) = 1$

Allm

$$\boxed{\begin{aligned} y &= {}^a\log(a^y) \\ a^{{}^a\log(x)} &= x \end{aligned}}$$

Ex $e^{\ln(x)} = x$, $x = 10^{\lg(x)}$

Viktiga lagar

- ${}^a\log(xy) = {}^a\log(x) + {}^a\log(y)$

  $\left(\ln(xy) = \ln(x) + \ln(y) \right)$

- ${}^a\log(x^\alpha) = \alpha\, {}^a\log(x)$

  $\left(\ln(x^\alpha) = \alpha \ln(x) \right)$

- ${}^a\log\left(\frac{x}{y}\right) = {}^a\log(x) - {}^a\log(y)$

  $\left(\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y) \right)$

- ${}^a\log(x) = \dfrac{{}^b\log(x)}{{}^b\log(a)}$ basbyte

Ex $\boxed{\lg(x) = \dfrac{\ln(x)}{\ln(10)}}$

$a = 10$

$b = e$

obs 1: $\ln(x+y) \neq \ln(x)+\ln(y)$

Ty, tag $x=e$ och $y=1$

$$\ln(x+y) = \ln(e+1)$$

$$\ln(x)+\ln(y) = \underbrace{\ln(e)}_{=1} + \underbrace{\ln(1)}_{=0} = 1$$

$$\therefore \ln(e+1) \neq 1$$

obs2! $\ln\left(\frac{x}{y}\right) \neq \frac{\ln(x)}{\ln(y)}$

Ty, Tag $x=e$, $y=1$

$$\ln\left(\frac{x}{y}\right) = \ln\left(\frac{e}{1}\right) = \ln(e) = 1$$

$$\frac{\ln(x)}{\ln(y)} = \frac{\ln(e)}{\ln(1)} = \frac{1}{0} = \infty$$

Ex: Lär in!

Lös $3^x = 2\cdot 10^{-x}$

Lösning Tag log av både led

$$\underbrace{\ln(3^x)}_{x\ln(3)} = \underbrace{\ln(2\cdot 10^{-x})}_{\ln(2)+\underbrace{\ln(10^{-x})}_{-x\ln(10)}}$$

$$\Rightarrow x\ln(3) = \ln(2) - x\ln(10)$$

$$\Rightarrow x\ln(3) + x\ln(10) = \ln(2)$$

$$\Rightarrow x\big(\underbrace{\ln(3)+\ln(10)}_{\underbrace{\ln(3\cdot 10)}_{\ln(30)}}\big) = \ln(2)$$

$$\Rightarrow x = \frac{\ln(2)}{\ln(30)} \neq \ln\left(\frac{2}{30}\right)$$

Ex2 Lär in

Lös $\ln(x^2+x-3) = \ln(x+1)$

Lösning om $\ln(A) = \ln(B)$

$\Longrightarrow A = B$ (Kontinuitet hos ln)

och skall man på slutet kolla de funna rötterna då $\ln(A)$ def endast då $A > 0$

Här $\ln(x^2+x-3) = \ln(x+1)$ $\circledast$

$$\Longrightarrow x^2+x-3 = x+1$$

$$\Longrightarrow x^2 = 4 \Longrightarrow x = \pm 2$$

Vi måste kolla båda lösningarna!

$x=2$: $\ln(\underbrace{2^2+2-3}_{3}) = \ln(2+1)$ sant

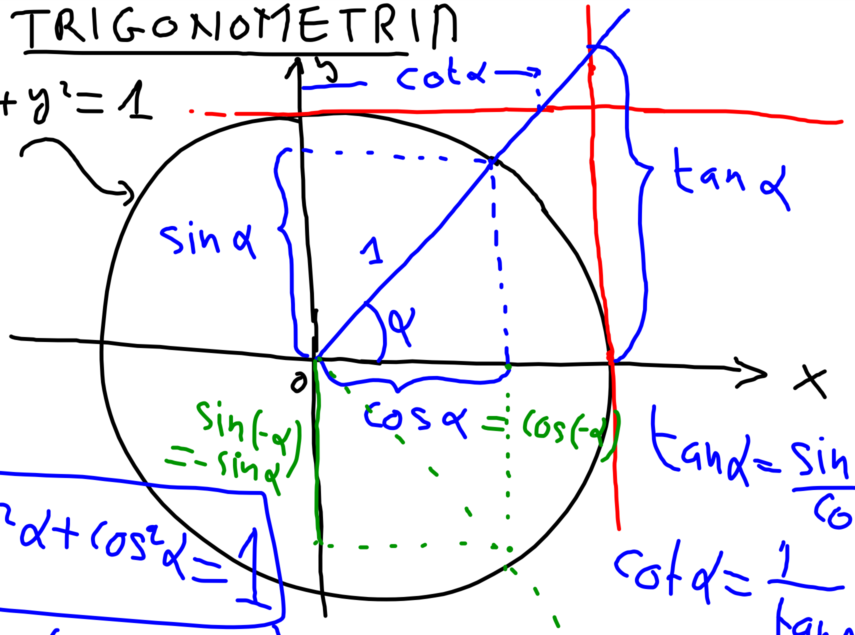
$x=-2$: $\ln(\underbrace{4-2-3}_{-1}) = \ln(\underbrace{-2+1}_{-1})$

$\ln(-1)$ ej def. gäller ej!

SVAR $x = 2$!

2. TRIGONOMETRIN

$x^2+y^2=1$



$$\boxed{\sin^2\alpha+\cos^2\alpha=1}$$

$$\tan\alpha=\frac{\sin\alpha}{\cos\alpha}$$

$$\cot\alpha=\frac{1}{\tan\alpha}$$

$\sin(\alpha+2n\pi)=\sin\alpha$, $n=0,\pm1,\pm2,\ldots$

$\cos(\alpha+2n\pi)=\cos\alpha$,

sin α och cos α har period 2π

Obs! $\sin(-\alpha)=-\sin\alpha$

$\cos(-\alpha)=+\cos\alpha$

$\tan(\alpha+n\pi)=\tan\alpha$, $n=0,\pm1,\ldots$

$\cot(\alpha+n\pi)=\cot\alpha$, $n=0,\pm1,\ldots$

tan α och cot α har period π

TRigonometriska Ekvationer

- $\sin(\underset{\substack{\uparrow \\ \text{okänd}}}{x}) = \sin(\underset{\substack{\uparrow \\ \text{känd}}}{a}) \Longrightarrow x = \begin{cases} a + n\cdot 2\pi \\ \pi - a + n\cdot 2\pi \end{cases}$ $n = 0, \pm 1, \pm \dots$
- $\cos(x) = \cos(a) \Longrightarrow x = \pm a + n\cdot 2\pi$ $n = 0, \pm 1, \pm 1 \dots$
- $\tan(x) = \tan(a) \Longrightarrow x = a + n\cdot\pi$ $n = 0, \pm 1, \pm 2, \dots$
- $\cot(x) = \cot(a) \Longrightarrow x = a + n\pi$ $n = 0, \pm 1, \dots$

ATT kunna väl

- $\cos^2(\alpha) + \sin^2(\alpha) = 1$
- $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

(Ex $\cos(2x) = \cos(x)\cos(x) - \sin(x)\sin(x)$
$= \cos^2(x) - \sin^2(x)$
$= \cos^2(x) - (1 - \cos^2(x))$
$= 2\cos^2(x) - 1$)

- $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$

($\sin(2x) = 2\sin(x)\cos(x)$)

Ex Lös $\sin(2x) = \sin(x)$

Lösning $\sin(2x) = 2\sin(x)\cos(x)$

Då fås $\sin(2x) = \sin(x) \Longrightarrow$

$$2\sin(x)\cos(x) = 2\sin(x)\tfrac{1}{2}$$

$$\Longrightarrow 2\sin(x)\left(\cos x - \tfrac{1}{2}\right) = 0$$

$\Longrightarrow$ **antigen** $\sin(x) = 0 \Longrightarrow x = n\pi$

$n = 0, \pm 1, \pm 2 \ldots$

eller $\cos(x) = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$

$$\Longrightarrow x = \pm\frac{\pi}{3} + n \cdot 2\pi, \quad n = 0, \pm 1, \ldots$$

SVAR $\sin(2x) = \sin(x)$ har lösningarna

$x = n\pi$ eller $x = \pm\frac{\pi}{3} + 2n\pi$

$n = 0, \pm 1, \pm 2, \ldots\ldots$

$$2\sin x \cos x = \sin x$$

$$\sin x(1 - 2\cos x) = 0$$

$\Longrightarrow$ antigen $\sin x = 0$

eller $1 - 2\cos x = 0$

$$\Longrightarrow \cos x = \frac{1}{2}$$

Binomialformeln: handlar om att räkna ut $(a+b)^n$, $n=0,1,2. \ldots.$

$$(a+b)^2 = a^2+2ab+b^2$$

$$(a+b)^3 = (a+b)(a+b)^2 = (a+b)(a^2+2ab+b^2)$$

$$= a^3+3a^2b+3ab^2+b^3$$

Hur räknar man

$(a+b)^n$, n god post heltal

Def 1 $n! = 1\cdot 2\cdot 3\cdot 4\cdots(n-1)n$

$= (n-1)!\,n$

$$\boxed{0! = 1! = 1}$$

Vad innebär $n!$

$n! =$ antalet sätt att sortera n element i en viss ordning.

$=$ antalet "password" utan rep av n olika element

Ex $M = \{1, 2, 3\}$

antalet utan rep passw $3! = 1\cdot 2\cdot 3 = 6$

123, 213, 312, 132, 321, 231

Def 2 $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$ (n över k)
$n > k$

= antalet delmängder av storlek k ur en mängd av storlek n.

Ex 3 tennis spelare 1, 2, 3 skall spela mot varan! Hur många matcher!

Här har vi $n=3$, $k=2$

$$\binom{n}{k} = \binom{3}{2} = \frac{3!}{(3-2)!\,2!} = \frac{6}{1!\,2} = 3!$$

Räknelagar

1º $\binom{n}{0} = \binom{n}{n} = 1$

2º $\binom{n}{n-k} = \binom{n}{k}$

3º $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$

Kan lätt kontrolleras via

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

Binomial formeln

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2$$

$$+ \dots \binom{n}{n-1} a\, b^{n-1} + \binom{n}{n} b^n$$

Ex Finn koeff framför x^{13} i polynomet $(x+1)^{15}$

Lösning Använd binomialformeln

$$(\underset{\underset{a}{\uparrow}}{x}+\underset{\underset{b}{\uparrow}}{1})^{15} = \sum_{k=0}^{15} \binom{15}{k} \underset{\underset{a}{\uparrow}}{x}^{15-k}\, \underset{\underset{b}{\uparrow}}{1}^{k}$$

$$= \sum_{k=0}^{15} \underbrace{\binom{15}{k}}_{\text{koefficient}} \overset{\text{framför}}{x^{15-k}}$$

koeff framför x^{15-k} är $\binom{15}{k}$

vill ha " " x^{13} (välj $k=2$)

$$= \binom{15}{2} = \frac{15!}{(15-2)!\,2!} = \frac{15!}{13!\,2!}$$

$$= \frac{\cancel{13!}\,14\cdot 15}{\cancel{13!}\,\underset{=2}{2!}} = \frac{14\cdot 15}{2} = 7\cdot 15$$

$$= 105$$

Svar 105

BRA EX

Visa att $$\sum_{k=0}^{n}\binom{n}{k}=2^n$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0$$

Lösning

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

$$(\underbrace{1+1}_{2})^n=\sum_{k=0}^{n}\binom{n}{4}\underbrace{1^{n-k}}_{1}\underbrace{1^{k}}_{=1}$$

$$2^n=\sum_{k=0}^{n}\binom{n}{k}$$

$$(\underbrace{1-1}_{=0})^n=\sum_{k=0}^{n}\binom{n}{k}\underbrace{1^{n-k}}_{=1}(-1)^k$$

$$\therefore 0=\sum_{k=0}^{n}(-1)^k\binom{n}{k}$$