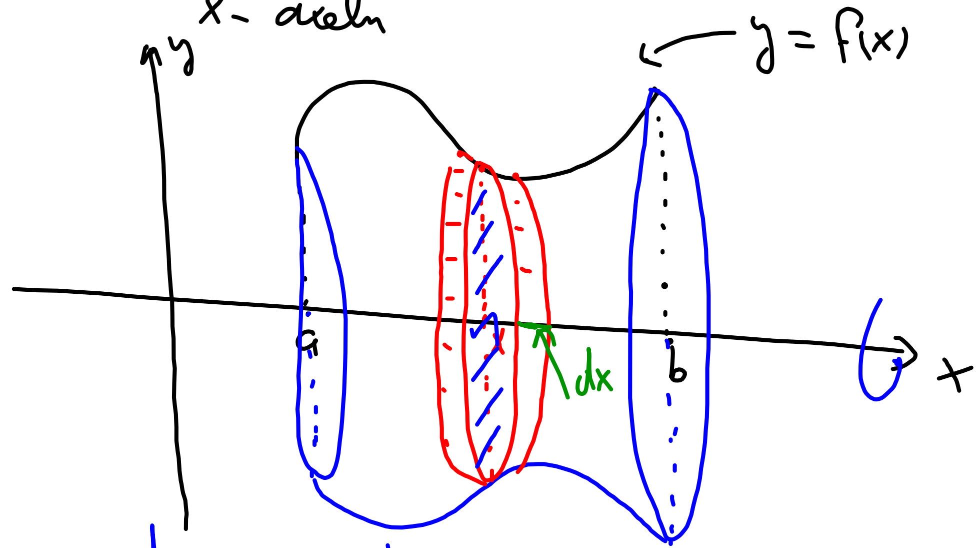
# Forel 18 Rotationsvolymberäkningar

- $y = f(x)$, $a \le x \le b$ roterar omkring $x$-axeln
- $y = f(x)$, $a \le x \le b$ roterar omkring $y$-axeln

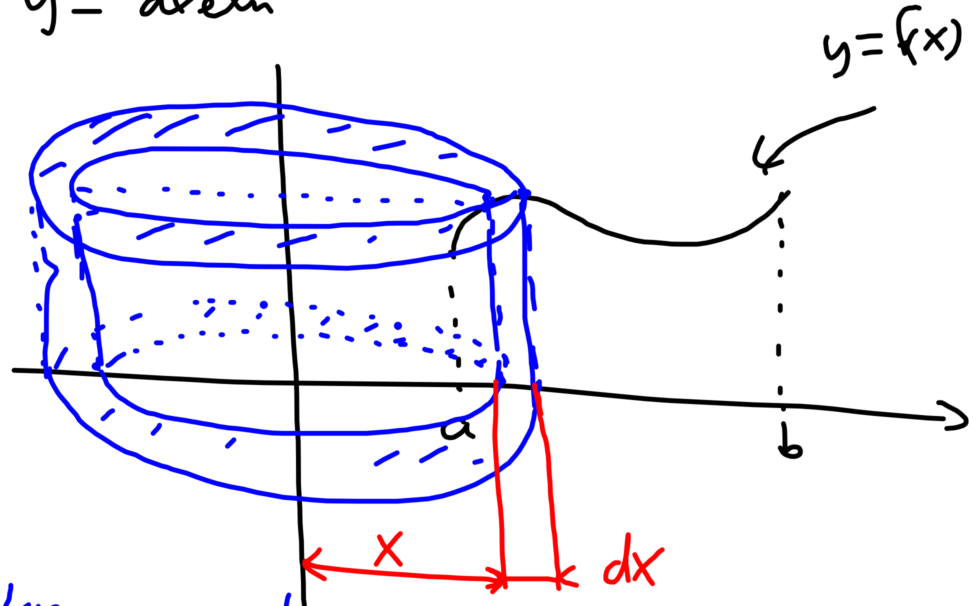
1. $y = f(x)$, $a \le x \le b$, roterar omkring $x$-axeln




$$dV(x) = \underbrace{\text{basarean}}_{\pi \underbrace{r^2(x)}_{f^2(x)}} \times \underbrace{\text{tjocklek}}_{dx}$$

$$V_x = \int_a^b dV(x) = \pi \int_a^b f^2(x)\,dx \ge 0$$

2. $y = f(x)$, $a \le x \le b$ roterar omkring y-axeln




Vi beräknar volymen mha sk rör element

Volymen för röret

$$dV(x) = \underbrace{\text{Omkrets}}_{2\pi x} \cdot \underbrace{\text{höjden}}_{f(x)} \cdot \underbrace{\text{tjocklek}}_{dx}$$

$$V_y = \int_a^b dV(x) = 2\pi \int_a^b |x\, f(x)|\, dx \ge 0$$

Ex $D = \{(x,y): 1 \le x \le 2,\ y = \frac{1}{x\sqrt{x+1}}\}$

Beräkna Volymen i följande fall

① D roterar omkring x-axeln

② D — " — y-axeln

Lösning ① Volymen ges då av

$$V_x = \pi \int_a^b f^2(x)\,dx = \pi \int_1^2 \left(\frac{1}{x\sqrt{x+1}}\right)^2 dx =$$

$$= \pi \int_1^2 \frac{1}{x^2(x+1)}\,dx = \pi I$$

Vi räknar ut $\int_1^2 \frac{1}{x^2(x+1)}\,dx$

Ansats:

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$= \frac{A(x(x+1)) + B(x+1) + Cx^2}{x^2(x+1)}$$

$$= \frac{A(x^2+x) + B(x+1) + Cx^2}{x^2(x+1)}$$

Sant om

$$1 = A(x^2+x) + B(x+1) + Cx^2$$

$$1 = x^2(A+C) + x(A+B) + B$$

$$\Longrightarrow \begin{cases} 1 = B \\ 0 = A + C \Rightarrow C = -A = 1 \\ 0 = A + B \Rightarrow A = -B = -1 \end{cases}$$

$$\therefore \frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}$$

$$\int_1^2 \frac{1}{x^2(x+1)}\,dx = -\int_1^2 \frac{1}{x}\,dx + \int_1^2 \frac{1}{x^2}\,dx + \int_1^2 \frac{1}{x+1}\,dx$$

$$= -\left[\ln x\right]_1^2 + \left[\frac{-1}{x}\right]_1^2 + \left[\ln(x+1)\right]_1^2$$

$$= -\underbrace{(\ln 2 - \underbrace{\ln 1}_{=0})}_{\ln 2} - \underbrace{\left(\underbrace{\frac{1}{2} - \frac{1}{1}}_{=-1/2}\right)}_{1/2} + \ln(3) - \ln(2)$$

$$= -\ln 2 + 1/2 + \ln 3 - \ln 2$$

$$= \underbrace{\ln 3 - \underbrace{2\ln 2}_{\ln 4}}_{\ln\left(\frac{3}{4}\right)} + 1/2$$

$$\int_1^2 \frac{1}{x^2(x+1)}\,dx = \ln\left(\frac{3}{4}\right) + 1/2$$

SVAR

$$V_x = \pi \int_1^2 f^2(x)\,dx = \pi\left(\ln\left(\frac{3}{4}\right) + \frac{1}{2}\right)$$

Volym då D roterar omkring y-axeln

$$V_y = \int_1^2 2\pi x f(x)\,dx = 2\pi \int_1^2 \frac{\cancel{x}}{\cancel{x}} \cdot \frac{1}{\sqrt{x+1}}\,dx$$

$$= 2\pi \int_1^2 \frac{1}{\sqrt{x+1}}\,dx = \left[\begin{array}{l} t = x+1 \\ \frac{dt}{dx} = 1 \Leftrightarrow dt = dx \\ x: 1 \longrightarrow 2 \\ t: \underbrace{1+1}_{2} \longrightarrow \underbrace{2+1}_{3} \end{array}\right] =$$

$$= 2\pi \int_2^3 \frac{1}{\sqrt{t}}\,dt =$$

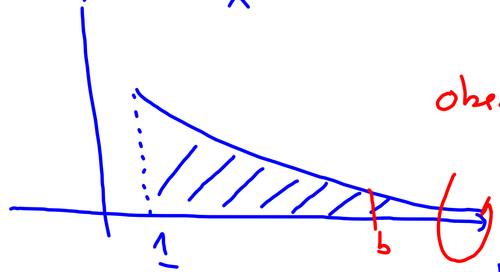
$$= 2\pi \int_2^3 t^{-1/2}\,dt = 2\pi \left[\frac{t^{-1/2+1}}{-1/2+1}\right]_2^3$$

$$= 2\pi \left[\frac{\sqrt{t}}{1/2}\right]_2^3 = 4\pi\left(\sqrt{3}-\sqrt{2}\right) > 0$$

<u>SVAR</u> $\int_1^2 2\pi x f(x)\,dx = 4\pi\left(\sqrt{3}-\sqrt{2}\right) > 0$

# ATT FÖRSTÅ Konvergens – Divergens AV Generaliserade Integraler

**EX1** $f(x) = \frac{1}{x}$ , $x \geq 1$




obegränd AREA

begränsad volym

(1) Arean $= \int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x}\,dx =$

$$= \lim_{b\to\infty} \Big[\ln x\Big]_1^b = \lim_{b\to\infty} \ln b - \underbrace{\ln 1}_{=0} = \infty$$

(2) Låt ytan $D: \{(x,y),\ 1 \leq x < \infty,\ y = \frac{1}{x}\}$ roteras ett varv kring x-axeln

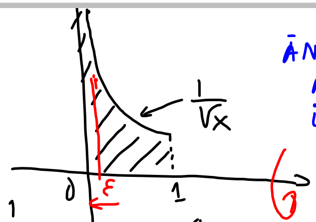
$$V_x = \int_1^{\infty} \pi f^2(x)\,dx = \pi \int_1^{\infty} \left(\frac{1}{x}\right)^2 dx =$$

$$= \pi \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2}\,dx = \pi \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b =$$

$$= \pi \lim_{b\to\infty} \left(-\frac{1}{b} - \left(-\frac{1}{1}\right)\right) = \pi \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right)$$

$$= \pi(0+1) = \pi$$

EX2




ÄNDLIG AREAN
icke ändlig volym

Arean: $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \int_0^1 x^{-1/2}\,dx =$

$= \lim_{\varepsilon\to 0+} \int_\varepsilon^1 x^{-1/2}\,dx = \lim_{\varepsilon\to 0+} \left[\frac{x^{-1/2+1}}{-1/2+1}\right]_\varepsilon^1$

$= \lim_{\varepsilon\to 0+} \left[2\sqrt{x}\right]_\varepsilon^1 = 2\sqrt{1} - \underbrace{\lim_{\varepsilon\to 0+} 2\sqrt{\varepsilon}}_{=0}$

$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$

Volymen då $\{(x,y) : 0 \le x \le 1,\ y = \frac{1}{\sqrt{x}}\}$
Roterar omkring x-axeln

$V_x = \int_0^1 \pi f^2(x)\,dx = \pi \int_0^1 \left(\frac{1}{\sqrt{x}}\right)^2 dx$

$= \pi \int_0^1 \frac{1}{x}\,dx = \pi \lim_{\varepsilon\to 0+} \int_\varepsilon^1 \frac{1}{x}\,dx$

$= \pi \lim_{\varepsilon\to 0+} \left[\ln x\right]_\varepsilon^1 = \pi \lim_{\varepsilon\to 0+} (\underbrace{\ln 1}_{=0} - \ln\varepsilon)$

$= \pi \underbrace{\lim_{\varepsilon\to 0+} -\ln\varepsilon}_{\infty}$

$= \infty$