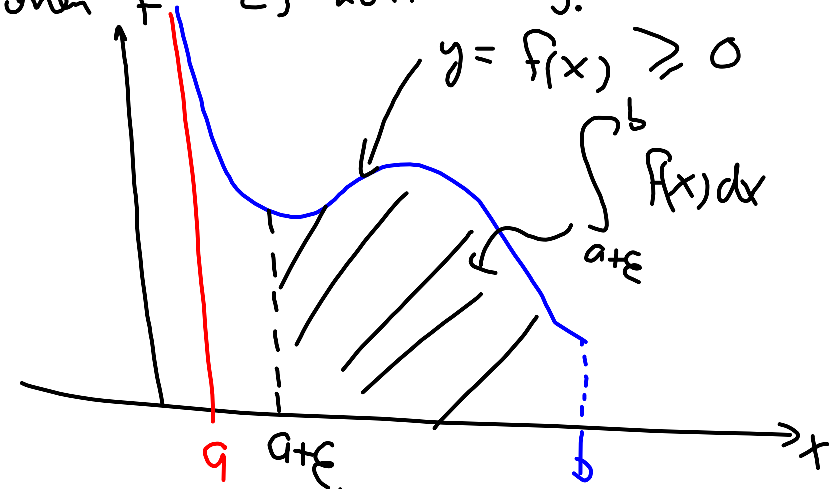
Förel 17. Generaliserade integraler med tillämpningar

Fallet då intervallet (a,b) ändligt men funktionen f Ej kontinuerlig.



$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^{b} f(x)\,dx = \lim_{\varepsilon \to 0+} I(\varepsilon)$$

då $I(\varepsilon) = \int_{a+\varepsilon}^{b} f(x)\,dx$

ATT kunna

$$\int_0^1 \frac{1}{x^\alpha}\,dx = \begin{cases} \frac{1}{1-\alpha}, & \alpha < 1 \\ \infty, & \alpha \geq 1 \end{cases}$$

Lösning $\displaystyle\int_0^1 \frac{1}{x^\alpha}\,dx = \lim_{\varepsilon\to 0+}\int_\varepsilon^1 \frac{1}{x^\alpha}\,dx =$

$$= \lim_{\varepsilon\to 0+} I(\varepsilon)$$

$$I(\varepsilon) = \int_\varepsilon^1 x^{-\alpha}\,dx = \left[\frac{x^{-\alpha+1}}{1-\alpha}\right]_\varepsilon^1 =$$

$\alpha \neq 1$

$$= \frac{1}{1-\alpha}\left(\underbrace{1^{-\alpha+1}}_{=1} - \varepsilon^{1-\alpha}\right) =$$

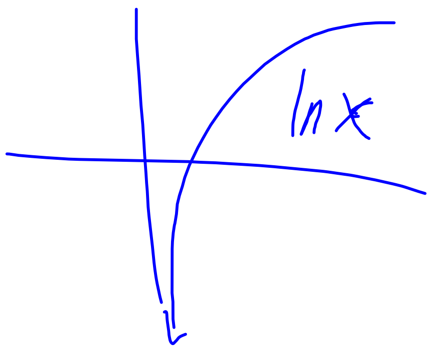
$$= \frac{1}{1-\alpha}\left(1 - \varepsilon^{1-\alpha}\right)$$

$= \Big[\ \varepsilon^{1-\alpha} \xrightarrow[\varepsilon\to 0+]{} 0$ då $1-\alpha>0 \iff 1>\alpha \iff \alpha<1$

$\varepsilon^{1-\alpha} \xrightarrow{1-\alpha<0} \infty$ då $\alpha>1$

$\alpha=1$: $\displaystyle\int_0^1 \frac{1}{x}\,dx = \lim_{\varepsilon\to 0+}\int_\varepsilon^1 \frac{1}{x}\,dx = \lim_{\varepsilon\to 0+}\left(\ln(1) - \ln(\varepsilon)\right)$

$$= 1 - (-\infty) = \infty$$



Ex $\int_0^\infty \frac{1}{x^\alpha}\,dx$ divergerar (konvergerar inte) för alla $\alpha \in \mathbb{R}$

Ty $\int_0^\infty \frac{1}{x^\alpha}\,dx = \underbrace{\underbrace{\int_0^1 \frac{1}{x^\alpha}\,dx}_{= \begin{cases} \frac{1}{1-\alpha}, & \alpha<1 \\ \infty, & \alpha\geq 1 \end{cases}} + \underbrace{\int_1^\infty \frac{1}{x^\alpha}\,dx}_{= \begin{cases} \frac{1}{\alpha-1}, & \alpha>1 \\ \infty, & \alpha\leq 1 \end{cases}}}_{\text{finns ingen gemen. } \alpha \text{ : Integralerna är ändliga}}$

Def $\underbrace{\int_a^\infty f(x)\,dx}_{\text{konv}} = \int_a^b f(x)\,dx + \int_b^\infty f(x)\,dx$

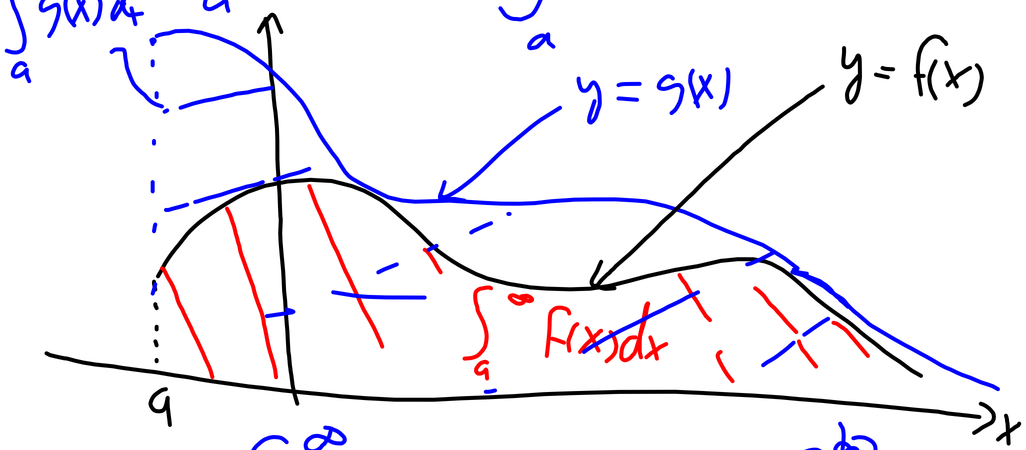
konv $\Longleftrightarrow$ $\underbrace{\int_a^b f(x)\,dx}_{\text{konv}} + \underbrace{\int_b^\infty f(x)\,dx}_{\text{konv.}}$

Jämförelsekriterium för generaliserade Integraler

Låt $0 \le f(x) \le g(x)$ $\quad x \geq a$

Då gäller att

$$\int_a^\infty f(x)\,dx \le \int_a^\infty g(x)\,dx$$



a) om $\int_a^\infty g(x)\,dx$ ändlig $\Rightarrow \int_a^\infty f(x)\,dx$ ändlig

b) om $\int_a^\infty f(x)\,dx$ icke-ändlig $\Rightarrow$

$\int_a^\infty g(x)\,dx$ icke-ändlig

Ex Undersök om $\int_1^\infty \frac{1}{x^2+\ln x}\,dx$ är konv.

Lösning

$$0 \le f(x) = \frac{1}{x^2+\ln x} \le \frac{1}{x^2} = g(x)$$

$$\Rightarrow \int_1^\infty \frac{1}{x^2+\ln x}\,dx \le \int_1^\infty \frac{1}{x^2}\,dx = 1$$

$$\int_1^\infty \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^\infty = 0-\left(-1\right) = 1$$

$\therefore \int_1^\infty \frac{1}{x^2+\ln x}\,dx$ är konvergent

Ex Beräkna $\int_0^\infty xe^{-x}dx$

Lösning $\int_0^\infty xe^{-x}dx = \lim_{b\to\infty} \int_0^b xe^{-x}dx$

$\int_0^b xe^{-x}dx = \left[\int u'v\,dx = uv - \int uv'\,dx \right]$

$= \left[x(-e^{-x})\right]_0^b - \underbrace{\int_0^b (-e^{-x})\,1\,dx}_{\left[e^{-x}\right]_0^b} =$

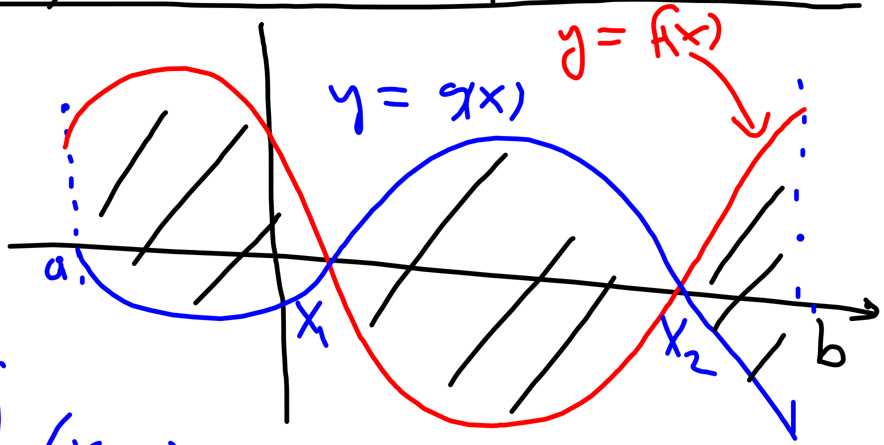
$= \left[x(-e^{-x})\right]_0^b - \left[e^{-x}\right]_0^b$

$= -\left[e^{-x}(x+1)\right]_0^b = 1 - e^{-b}(b+1)$

$\lim_{b\to\infty} e^{-b}(b+1) = 0$

$\therefore \int_0^\infty xe^{-x}dx = \lim_{b\to\infty}\left(1 - e^{-b}(b+1)\right) = 1$

Beräkning av areor av plana områden



$D = \{ (x, y),\ a \le x \le b$ och y begränsas av kurvana $y = f(x)$ och $y = g(x)$

Arean av D : $\int_a^b |f(x) - g(x)|\,dx \geq 0$

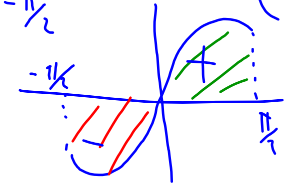
$$\int_a^b |f(x) - g(x)|\,dx = \int_a^{x_1} (f(x) - g(x))\,dx + \int_{x_1}^{x_2} (g(x) - f(x))\,dx + \int_{x_2}^{b} (f(x) - g(x))\,dx$$

Ex Hur mycket färg behövs för att måla området

$$D: \left\{ (x,y)\ -\frac{\pi}{2} \le x \le \frac{\pi}{2},\ y = \sin x \right\}$$

Vanligt fel

$$\int_{-\pi/2}^{\pi/2} \sin x \, dx = \Big[-\cos x\Big]_{-\pi/2}^{\pi/2} = 0$$



$$\int_{-\pi/2}^{\pi/2} |\sin x| \, dx = \underbrace{\int_{-\pi/2}^{0} -\sin x \, dx}_{\int_0^{\pi/2} \sin x \, dx} + \int_{0}^{\pi/2} \sin x \, dx$$

$$= 2\int_{0}^{\pi/2} \sin x \, dx = 2\Big[-\cos x\Big]_{0}^{\pi/2} =$$

$$= +2\underbrace{\Big(\underbrace{-\cos \tfrac{\pi}{2}}_{=0} - \big(-\underbrace{\cos 0}_{=1}\big)\Big)}_{=1}$$

$$= 2$$

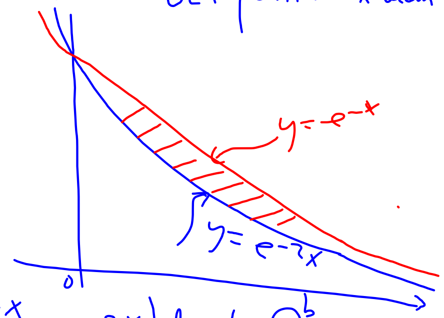
$$\int_{-\pi/2}^{0} -\sin x \, dx = \left[\begin{array}{l} x = -t \\ \sin x = \sin(-t) = -\sin t \\ dx = -dt \\ x: -\frac{\pi}{2} \to 0 \\ t: \frac{\pi}{2} \to 0 \end{array}\right]$$

$$= \int_{\pi/2}^{0} +(+\sin t)(-dt) = -\int_{\pi/2}^{0} \sin t \, dt$$

$$= -\Big(-\int_{0}^{\pi/2} \sin t \, dt\Big)$$

Ex Beräkna arean mellan $y = f(x) = e^{-x}$ och $y = g(x) = e^{-2x}$ och positiva x-axeln

Lösning



$$\int_0^{\infty} \left(e^{-x} - e^{-2x}\right) dx = \lim_{b \to \infty} \int_0^b \left(e^{-x} - e^{-2x}\right) dx$$

$$= \lim_{b \to \infty} \left[-e^{-x} + \frac{e^{-2x}}{+2}\right]_0^b = \lim_{b \to \infty} \left[\frac{e^{-2x}}{2} - e^{-x}\right]_0^b$$

$$= \lim_{b \to \infty} \left(\frac{e^{-2b}}{2} - e^{-b} - \underbrace{\left(\frac{e^0}{2} - e^0\right)}_{-1/2}\right)$$

$$= \lim_{b \to \infty} \left(\frac{e^{-2b}}{2} - e^{-b} + 1/2\right) = 0 + 1/2$$

$$\int_0^{\infty} \left(e^{-x} - e^{-2x}\right) dx = 1/2 > 0$$

Obs! Fråga

$$e^{-2x} = e^{-x} \iff x = 0$$

$$\int_0^{\infty} \left(e^{-2x} - e^{-x}\right) dx \stackrel{!}{=} -1/2$$

Skriv oj då! Har missat ngt det skall vara positivt!

Svar $\int_0^{\infty} \left(e^{-x} - e^{-2x}\right) dx = 1/2$ ☺

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