Förel 15: • Exempel hur man integrera rationella funkter
• Grundsats för primitiva fkner

EX1. $\int \frac{x^3+1}{x^2+7x+12}\,dx = \int \frac{P(x)}{Q(x)}\,dx$

har grad $P = 3 >$ grad $Q = 2$

steg 1 Division

$$\frac{P(x)}{Q(x)} = \frac{x^3+1}{x^2+7x+12} = \underbrace{x-7}_{q(x)} + \frac{37x+85}{x^2+7x+12} = \frac{R(x)}{Q(x)}$$

$$\int \frac{x^3+1}{x^2+7x+12}\,dx = \underbrace{\int (x-7)\,dx}_{\frac{x^2}{2}-7x+C_1} + \underbrace{\int \frac{37x+85}{x^2+7x+12}\,dx}_{I_1}$$

steg 2 $\int \frac{37x+85}{x^2+7x+12}\,dx$

$x^2+7x+12 = 0 \implies x = -4, -3$

$\implies x^2+7x+12 = (x+4)(x+3)$

$$\int \frac{37x+85}{x^2+7x+12}\,dx = \int \frac{A}{x+4}\,dx + \int \frac{B}{x+3}\,dx$$

$$\frac{37x+85}{x^2+7x+12} = \frac{A}{x+4} + \frac{B}{x+3} = \frac{A(x+3)+B(x+4)}{(x+4)(x+3)}$$

Sant om $37x+85 = (A+B)x + 3A+4B$

$$\implies \begin{cases} 37 = A+B \\ 85 = 3A+4B \end{cases} \implies \begin{cases} A = 63 \\ B = -26 \end{cases}$$

Vi får då:

$$\int \frac{37x+85}{x^2+7x+12}\,dx = 63\int \frac{1}{x+4}\,dx - 26\int \frac{1}{x+3}\,dx$$

$$= 63\ln|x+4| - 26\ln|x+3| + C_2$$

steg 3 Skriv svaret

$$\int \frac{x^3+1}{x^2+7x+12}\,dx = \frac{x^2}{2} - 7x + 63\ln|x+4| - 26\ln|x+3| + \text{Konstant}$$

Ex2 $\int \frac{x+1}{x^2-2x+2}\,dt$

1) kolla om x^2-2x+2 har reella rötter

$x^2-2x+2 = \underbrace{x^2-2x+1}_{(x-1)^2}+1 = (x-1)^2+1 > 0$

Inga Reella rötter.

2) $\int \frac{x+1}{x^2-2x+2}\,dt = \int \frac{x+1}{(x-1)^2+1}\,dt =$

$= \left[\begin{array}{l} t = x-1 \Leftrightarrow x = t+1 \\ \frac{dt}{dx} = 1 \Leftrightarrow dt = dx \end{array}\right] = \int \frac{t+1+1}{t^2+1}\,dt$

$= \int \frac{t+2}{t^2+1}\,dt = 2\underbrace{\int \frac{1}{t^2+1}\,dt}_{\arctan t + C_1} + \int \frac{t}{t^2+1}\,dt$

$= 2\arctan t + 2C_1 + \frac{1}{2}\int \frac{2t}{t^2+1}\,dt$

[obs! $\frac{d}{dt}(t^2+1) = 2t$ jämför med täljaren

$\int \frac{u'(t)}{u(t)}\,dt = \ln|u(t)| + C$]

$= 2\arctan t + \frac{1}{2}\ln(t^2+1) + \text{konstant}$

$= [t = x-1] =$

$= 2\arctan(x-1) + \frac{1}{2}\ln\left((x-1)^2+1\right) + \text{konst}$

SVAR $\int \frac{x+1}{x^2-2x+2}\,dt = 2\arctan(x-1) +$ minus

$+ \frac{1}{2}\ln(x^2-2x+2) + \text{konst}$

Grundsatsen för primitiva fkn.

eller När en funktion f har en primitiv fkn F

Antag (a) f kontinuerlig $a \le x \le b$

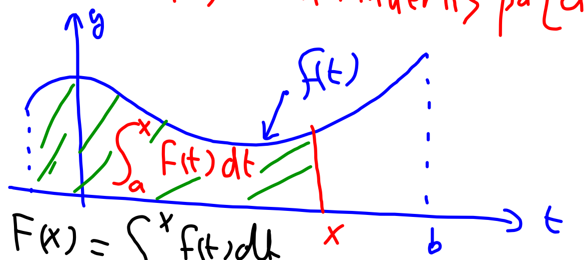
(b) Sätt $F(x) = \int_a^x f(t)\,dt$

Då gäller

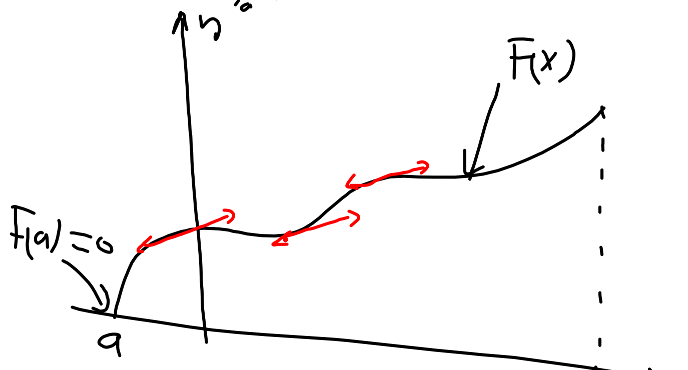
(1) $$\frac{dF}{dx} = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

(2) $F(x)$ kontinuerlig på $[a, b]$

En bild



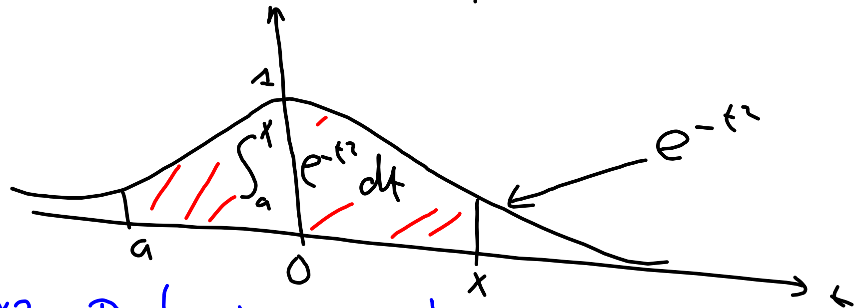
$$\begin{cases} F(x) = \int_a^x f(t)\,dt \\ F(a) = \int_a^a f(t)\,dt = 0 \end{cases}$$



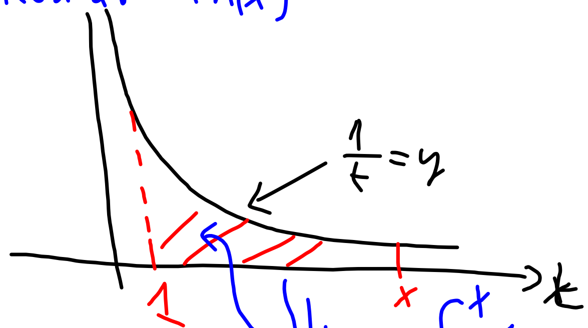
En partikel rör sig på banan $F(x)$, $a \le x \le b$, med start i $F(a) = 0$ med hastighet $F'(x) = f(x)$

EX1 $f(x) = e^{-x^2}$

$$F(x) = \int_a^x e^{-t^2}\,dt$$ = Vi kan Ej EXPLICIT en primitiv fkn



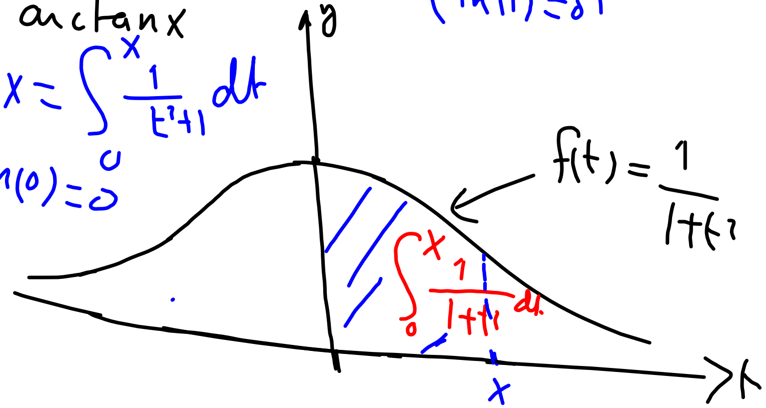
EX2 Definition av $\ln(x)$



$$\begin{cases} \ln x = \int^x \frac{1}{t}\,dt \\ \ln(1) = 0 \end{cases}$$

EX3 Def av arctan x

$$\begin{cases} \arctan x = \int_0^x \frac{1}{t^2+1}\,dt \\ \arctan(0) = 0 \end{cases}$$



Tillämpning av kedjeregeln $\frac{d}{dx} f[u(x)] = \frac{df}{du}\frac{du}{dx}$

- $f(t)$ kontinuerlig $u(x) \le t \le v(x)$
- $u'(x), v'(x)$ finns på $a < x < b$

$$\Rightarrow \frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\,dt\right] = f(v(x))\,v'(x) - f(u(x))\,u'(x)$$

Ty Låt $F(t)$ vara en primitiv fkn till $f(t)$ dvs $F'(t) = f(t)$

$$\int_{u(x)}^{v(x)} f(t)\,dt = \Big[F(t)\Big]_{u(x)}^{v(x)} = F(v(x)) - F(u(x))$$

$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\,dt\right] = \underbrace{\frac{d}{dx} F(v(x))}_{=F'(v(x))v'(x)} - \underbrace{\frac{d}{dx} F(u(x))}_{F'(u(x))u'(x)}$$

$F' = f$ $\qquad$ $F' = f$

$$= f(v(x))\,v'(x) - f(u(x))\,u'(x)$$

BRA EX

$$\frac{d}{dx}\int_{x^2}^{x^3} e^{-t^2}dt = \left[\text{Fel!}\ \int e^{-t^2}dt = \frac{e^{-t^2}}{-2t}+c = \frac{d}{dx}\left(\frac{e^{-t^2}}{-2t}+c\right)\right.$$

$$\left. = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v-v'u}{v^2}\right] = \frac{\cancel{-2t}\,e^{-t^2}}{\cancel{-2t}}$$

Lösning: Låt $F(t)$ vara en primitiv till $f(t) = e^{-t^2}$ dvs $F'(t) = e^{-t^2}$

$$\int_{x^2}^{x^3} e^{-t^2}dt = \left[F(t)\right]_{x^2}^{x^3} = \underbrace{F(x^3)-F(x^2)}_{G(x)}$$

$$\Rightarrow \frac{d}{dx}\int_{x^2}^{x^3} e^{-t^2}dt = G'(x) = \frac{d}{dx}\left(F(x^3)-F(x^2)\right)$$

$$= \frac{d}{dx}F(x^3) - \frac{d}{dx}F(x^2) = \left[\frac{d}{dx}\left(F(u(x))\right) = F'(u(x))\,u'(x) = f(u(x))\,u'(x)\right] =$$

$$= F'(x^3)\frac{d}{dx}x^3 - F'(x^2)\frac{d}{dx}x^2$$

$$= f(x^3)\,3x^2 - f(x^2)\,2x$$

$$= \left[f(t) = e^{-t^2} \Rightarrow f(x^3) = e^{-(x^3)^2},\ f(x^2) = e^{-(x^2)^2}\right] =$$

$$= e^{-(x^3)^2}3x^2 - e^{-(x^2)^2}2x$$

$$= e^{-x^6}3x^2 - e^{-x^4}2x$$

$$= xe^{-x^4}\left(e^{-x^2}\cdot 3x - 2\right)$$

SVAR:

$$\frac{d}{dx}\int_{x^2}^{x^3} e^{-t^2}dt = xe^{-x^4}\left(3xe^{-x^2}-2\right)$$

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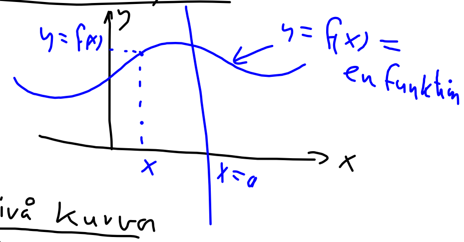
Geomtriska Tillämpningar av Integraler

Plana kurvor

Olika sätt att beskriva plana kurvor.

A. Funktionsgrafer



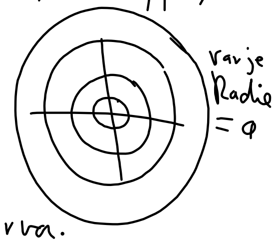
B. Nivå kurva

Alla punkter (x,y) som uppfyller

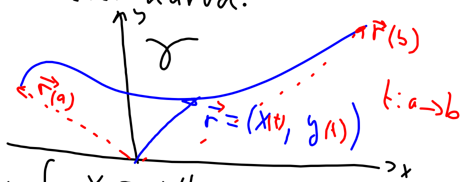
$F(x,y) = C$

Ex $x^2+y^2=a^2$

$x^2-y^2=1$



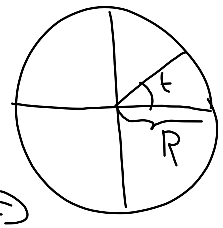
C. Parameter kurva.



$$\gamma : \begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad t: a \to b$$

Ex $x^2+y^2=R^2$

$$\begin{cases} x = R\cos t \\ y = R\sin t \end{cases}$$



Ex $y = f(x) \iff$

$$\begin{cases} x = t \\ y = f(t) \end{cases}$$