Förel 14: Substion via trigonometriska funktionen.

$\cos^2 x + \sin^2 x = 1$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\sin(x+y) = \sin x \cos y + \sin y \cos x$

$\Rightarrow \cos(2x) = \cos^2 x - \sin^2 x \quad (1)$

$\Rightarrow \cos^2 x = \cos 2x + \sin^2 x =$

$= \cos 2x + (1 - \cos^2 x)$

$\Rightarrow \boxed{\cos^2 x = \frac{\cos 2x + 1}{2}}$

Ur (1) $\Rightarrow \sin^2 x = \underbrace{\cos^2 x}_{1-\sin^2 x} - \cos 2x$

$= -\sin^2 x + 1 - \cos 2x$

$\Rightarrow \boxed{\sin^2 x = \frac{1 - \cos 2x}{2}}$

EX1 $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx =$

$= \frac{1}{2}\int 1\,dx - \frac{1}{2}\int \cos 2x\,dx$

$= \frac{1}{2}x + C_1 - \frac{1}{2}\,\frac{\sin 2x}{2} + C_2$

$\therefore \int \sin^2 x\,dx = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + \underbrace{C}_{C_1+C_2}$

P.S.S $\cos^2 x = \frac{1+\cos 2x}{2}$

$\int \cos^2 x\,dx = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C$

EX3 $\int \sin^4 x \cos^5 x\,dx =$

$= \int \sin^4 x \cos^4 x \cos x\,dx = \left[\begin{array}{l}\text{VET}\\ \frac{d}{dx}\cos x = -\sin x\end{array}\right]$

$\int \sin^4 x \,(\underbrace{1-\sin^2 x}_{\cos^2 x})^2 \cos x\,dx =$

$= \int \sin^4 x\,(1-\sin^2 x)^2 \underbrace{\cos x\,dx}_{\frac{d}{dx}\sin x} = \left[\begin{array}{l} t = \sin x \\ \frac{dt}{dx} = \cos x \\ \Rightarrow dt = \cos x\,dx \end{array}\right]$

$= \int t^4 (1-t^2)^2\,dt = \int t^4 (1+t^4-2t^2)\,dt =$

$= \int (t^4 + t^8 - 2t^6)\,dt = \frac{t^5}{5} + \frac{t^9}{9} - 2\frac{t^7}{7} + C$

$= [t = \sin x] = \frac{(\sin x)^5}{5} + \frac{(\sin x)^9}{9} - 2\,\frac{(\sin x)^7}{7} + C$

$= \int x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C$

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ATT bestämma $\int f(x)\,dx = F(x)+C$

då $f(x)$ innehåller $\sqrt{}$

Vi bör kunna!

(1) $$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

(2) $$\int \frac{u'(x)}{\sqrt{1-u^2(x)}}\,dx = \arcsin(u(x)) + C$$

(3) $$\int \frac{1}{\sqrt{a+x^2}}\,dx = \ln\left|x+\sqrt{x^2+a}\right| + C \qquad a \neq 0$$

EX 1 $\quad a>0 \quad \displaystyle\int \frac{1}{\sqrt{a-x^2}}\,dx =$

$$\sqrt{a-x^2} = \sqrt{a\left(1-\left(\frac{x}{\sqrt{a}}\right)^2\right)} = \sqrt{a}\,\sqrt{1-\left(\frac{x}{\sqrt{a}}\right)^2}$$

$\therefore a>0$ $$\int \frac{1}{\sqrt{a-x^2}}\,dx = \frac{1}{\sqrt{a}}\int \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{a}}\right)^2}}\,dx$$

$$= \left[\begin{array}{l} t = \frac{x}{\sqrt{a}} \\ dt = \frac{1}{\sqrt{a}}\,dx \Leftrightarrow dx = \sqrt{a}\,dt \end{array}\right] = \frac{1}{\cancel{\sqrt{a}}}\int \frac{\cancel{\sqrt{a}}\,dt}{\sqrt{1-t^2}}$$

$$= \int \frac{1}{\sqrt{1-t^2}}\,dt = \arcsin(t) + C$$

$$= \arcsin\left(\frac{x}{\sqrt{a}}\right) + C \qquad a>0$$

Ex
$$\int \frac{1}{\sqrt{9x^2-6x+2}}\,dx \sim \int \frac{1}{\sqrt{a+t^2}}\,dt = \ln\left|t+\sqrt{a+t^2}\right|+C$$

↑ positiv tecken

$9x^2-6x+2 =$ kvadratkompl.

$$(3x)^2-2(3x)+\underbrace{2}_{1+1} = \left[(a-b)^2 = a^2-2ab+b^2\right]$$

$$\underbrace{(3x)^2-2(3x)\cdot 1+1^2}_{(3x-1)^2}+1 = (3x-1)^2+1$$

$$\therefore \int \frac{1}{\sqrt{9x^2-6x+2}}\,dx = \int \frac{1}{\sqrt{1+(3x-1)^2}}\,dx$$

$$= \left[\int \frac{1}{\sqrt{a+t^2}}\,dt\right] = \left[\begin{array}{l} t = 3x-1 \\ dt = 3dx \iff \\ dx = \frac{1}{3}dt \end{array}\right] =$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{1+t^2}}\,dt = \frac{1}{3}\ln\left|t+\sqrt{1+t^2}\right|+C$$

$$= \frac{1}{3}\ln\left|(3x-1)+\sqrt{\underbrace{9x^2-6x+2}_{1+(3x-1)^2}}\right|+C$$

ATT integrera rationella integrander

1. $$\int \frac{1}{x+a}\,dx = \ln|x+a| + C$$

2. $$\int \frac{u'(x)}{u(x)}\,dx = \ln|u(x)| + C$$

3. $$\int \frac{1}{1+t^2}\,dt = \arctan(t) + C$$

4. $$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

5. $$\int \frac{u'(x)}{1+u^2(x)}\,dx = \arctan(u(x)) + C$$

Ex $\displaystyle\int \frac{x+2}{x^2+4x+6}\,dx = \left[\begin{array}{l} t = x^2+4x+6 \\ \frac{dt}{dx} = 2(x+2) \Rightarrow \\ \frac{dt}{2} = (x+2)\,dx \end{array}\right.$

$$= \int \frac{\frac{dt}{2}}{t} = \frac{1}{2}\int \frac{1}{t}\,dt = \frac{1}{2}\ln|t| + C$$

$$= \frac{1}{2}\ln|x^2+4x+6| + C$$

Ex $\int \frac{x+2}{x^2-3x+2}\,dx = \int f(x)\,dx$

steg 1 skriv om integranden $f(x)$

$$f(x) = \frac{x+2}{x^2-3x+2} = \left[\begin{matrix} x^2-3x+2=0 \\ x=1,2 \end{matrix}\right. ,\ \left. \begin{matrix} x^2-3x+2 = \\ (x-1)(x-2) \end{matrix}\right]$$

$$= \frac{x+2}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \quad (1)$$

steg 2 Finn A och B i (1) via Reducering på samma nämnaren

$$\frac{x+2}{(x-1)(x-2)} = \frac{A(x-2)}{(x-1)(x-2)} + \frac{B(x-1)}{(x-1)(x-2)} \quad (2)$$

steg 3 Identifiering av HL(2) = VL(2)

$$x+2 = \underbrace{A(x-2) + B(x-1)}_{Ax-2A+Bx-B}$$

$$x(A+B) - (2A+B)$$

$$1x + 2 = x(A+B) - (2A+B)$$

$$\Rightarrow \begin{matrix} 1 = A+B \\ 2 = -(2A+B) \end{matrix} \Rightarrow \begin{cases} A = -3 \\ B = 4 \end{cases}$$

kontrollera

steg 4

$$\frac{x+2}{(x-1)(x-2)} = \frac{-3}{x-1} + \frac{4}{x-2}$$

steg 5 integrera

$$\int \frac{x+2}{(x-1)(x-2)}\,dx = \underbrace{\int \frac{-3}{x-1}\,dx}_{-3\ln|x-1|+C_1} + \underbrace{\int \frac{4}{x-2}\,dx}_{4\ln|x-2|+C_2}$$

$$= -3\ln|x-1| + 4\ln|x-2| + \underbrace{C}_{C_1+C_2}$$

$$= \ln(x-1)^{-3} + \ln(x-2)^4$$

$$= \ln\frac{1}{(x-1)^3} + \ln(x-2)^4 + C$$

$$= \ln\left|\left(\frac{(x-2)^4}{(x-1)^3}\right)\right| + C$$

Integration av Rationella funktioner

dvs då integranden $f(x) = \frac{P(x)}{Q(x)} = \frac{\text{Polynom}}{\text{polynom}}$

Steg 1 Division

om grad $P(x) \geq$ grad $Q(x)$

$$f(x) = g(x) + \frac{R(x)}{Q(x)}, \quad \text{grad } R < \text{grad } Q$$

Integration

$$\int f(x)\,dx = \int g(x)\,dx + \int \frac{R(x)}{Q(x)}\,dx$$

Steg 2 att Ta fram $\int \frac{R(x)}{Q(x)}\,dx$ via

Partialbråks ansats

$Q(x)$	Ansats $\frac{R(x)}{Q(x)}$
$(x-a)(x-b)$	$\frac{A}{x-a} + \frac{B}{x-b}$
$(x-a)^2(x-b)$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
$(x-a)(x-b)(x-c)$	$= \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
$(x-a)\underbrace{(\alpha x^2+\beta)}_{\text{icke Reella lösningar}}$	$\frac{A}{x-a} + \frac{Bx+C}{\alpha x^2+\beta}$