Förel n:o 13 Fortsättning på Förel 12

Olika metod att finna primitiva fkner

A. Substitutionen = Gör ett variabels byte så att integranden "övergår" till en känd primitiv fkn

EX 1 . . . . .

EX 2: $\displaystyle\int \frac{x+1}{\sqrt{x^2+2x+3}}\,dx = \left[\begin{array}{l} t = x^2+2x+3 \\ \frac{dt}{dx} = 2(x+1) \Longrightarrow \\ dt = 2(x+1)\,dx \\ \Rightarrow \frac{dt}{2} = (x+1)\,dx \end{array}\right] =$

$$= \int \frac{\frac{1}{2}\,dt}{\sqrt{t}} = \frac{1}{2}\int t^{-1/2}\,dt = \frac{1}{2}\,\frac{t^{-1/2+1}}{-\frac{1}{2}+1} + C$$

$$= \cancel{\frac{1}{2}}\,\frac{t^{1/2}}{\cancel{\frac{1}{2}}} + C = \sqrt{t} + C = \left[t = x^2+2x+3\right]$$

Svar $\displaystyle\int \frac{x+1}{\sqrt{x^2+2x+3}}\,dx = \sqrt{x^2+2x+3} + C$

obs! "kontrollera alltid"

$$\frac{d}{dx}\left(\sqrt{x^2+2x+3} + C\right) = \frac{x+1}{\sqrt{x^2+2x+3}}$$

Ex3

$$\int \frac{x+1}{\sqrt{1-x^2}}\,dx = \left[\text{rya } \frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}} \right]$$

$$= \int \frac{x}{\sqrt{1-x^2}}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx = I_1 + I_2$$

$\Rightarrow$ a) $I_2 = \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$

b) $I_1 = \int \frac{x}{\sqrt{1-x^2}}\,dx = \left[\begin{array}{l} t = 1 - x^2 \\ \frac{dt}{dx} = -2x \Leftrightarrow dt = -2x\,dx \\ \Leftrightarrow -\frac{dt}{2} = +x\,dx \end{array} \right] =$

$$= \int \frac{-\frac{1}{2}dt}{\sqrt{t}} = -\frac{1}{2}\int t^{-1/2}\,dt =$$

$$= -\frac{1}{2}\,\frac{t^{-1/2+1}}{-1/2+1} + C = -\cancel{\frac{1}{1}}\,\frac{t^{1/2}}{+\cancel{1/2}} + C$$

$$= -\sqrt{t} + C = -\sqrt{1-x^2} + C$$

Svar $\int \frac{x+1}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2} + \arcsin x + K$

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Ex 4

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

a) $$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \left[\begin{array}{l} t = \cos x \\ \frac{dt}{dx} = -\sin x \\ dt = -\sin x\,dx \\ -dt = \sin x\,dx \end{array}\right] =$$

$$= \int -\frac{dt}{t} = -\int \frac{1}{t}\,dt = -\ln|t| + C$$

$$= [t = \cos x] = -\ln|\cos x| + C$$

Svar $$\int \tan x\,dx = -\ln|\cos x| + C$$

b) $$\int \frac{\cos x}{1+\sin x}\,dx = \left[\begin{array}{l} t = 1+\sin x \\ \frac{dt}{dx} = \cos x \Longleftrightarrow dt = \cos x\,dx \end{array}\right]$$

$$= \int \frac{dt}{t} = \ln|t| + C = \ln|1+\sin x| + C$$

Tillämpning $$\int \frac{u'(x)}{1+u^2(x)}\,dx = \arctan(u(x)) + C$$

att kunna: $(a \neq 0)$ $$\int \frac{1}{a^2+x^2}\,dx = \int \frac{1}{a^2\left(1+\left(\frac{x}{a}\right)^2\right)}\,dx$$

$$= \left[t = \frac{x}{a} \Rightarrow \frac{dt}{dx} = \frac{1}{a} \Longleftrightarrow a\,dt = dx \right] =$$

$$\int \frac{\cancel{a}^{1}\,dt}{a^{\cancel{2}}(1+t^2)} = \frac{1}{a}\int \frac{1}{1+t^2}\,dt = \frac{1}{a}\arctan t + C$$

$$= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C, \quad a \neq 0$$

SVAR $$\boxed{\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C}$$

b) $$\int \frac{\cos x}{1+\sin^2 x}\,dx = \left[\begin{array}{l} t = \sin x \\ \frac{dt}{dx} = \cos x \Longrightarrow dt = \cos x\,dx \end{array}\right]$$

$$= \int \frac{dt}{1+t^2} = \arctan t + C = \arctan(\sin x) + C$$

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Partiell integration $\iff (uv)' = u'v + v'u$

Vi $\quad (uv)' = u'v + v'u \iff$

$$u'v = (uv)' - v'u$$

$$\int u'(x)v(x)\,dx = \underbrace{\int \big(u(x)v(x)\big)'\,dx}_{u(x)v(x)} - \int v'(x)u(x)\,dx$$

$$\therefore \boxed{\int u'(x)v(x)\,dx = u(x)v(x) - \int v'(x)u(x)\,dx}$$

$$\boxed{\int f(x)g(x)\,dx = \big[F'(x) = f(x)\big] = F(x)g(x) - \int F(x)g'(x)\,dx}$$

Ex $\int \ln x\,dx = \int \underbrace{1}_{f(x)} \cdot \underbrace{\ln x}_{g(x)}\,dx = \left[\begin{array}{l} F(x) = x \\ g(x) = \ln x \\ g'(x) = \frac{1}{x} \end{array}\right]$

$= \underset{F(x)}{x}\,\underset{g(x)}{\ln x} - \int \underset{F}{x}\underbrace{\left(\frac{d}{dx}\ln x\right)}_{g'(x)} dx =$

$= x\ln x - \int \underbrace{\cancel{x} \cdot \frac{1}{\cancel{x}}}_{=1}\,dx =$

$= x\ln x - \int 1 \cdot dx = x\ln x - x + C$

$= x(\ln x - 1) + C$

Svar $\int \ln x\,dx = x(\ln x - 1) + C$

Ex2 $\int_0^1 \underbrace{x \arctan x}_{f(x)}\, dx = [F(x)]_0^1 = F(1) - F(0)$

Finn $\int x \arctan x \, dx = F(x) + C$

$$\int x \arctan x \, dx = \left[\begin{matrix} u'(x) = x & , v(x) = \arctan x \\ u(x) = \frac{x^2}{2} & v'(x) = \frac{1}{1+x^2} \end{matrix} \right]$$

$$= \left\{ \int u'(x) v(x) dx = u(x)v(x) - \int u(x) v'(x) dx \right\}$$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2} \int x^2 \cdot \frac{1}{1+x^2} dx =$$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2} \underbrace{\int \frac{x^2}{1+x^2} dx}_{I}$$

$$I = \int \frac{x^2}{1+x^2} dx = \int \frac{x^2+1-1}{x^2+1} dx$$

$$= \int \left(\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1} \right) dx = \int \left(1 - \frac{1}{1+x^2} \right) dx$$

$$= x - \arctan x + C_1$$

$$\int x \arctan x \, dx = \underbrace{\frac{x^2 \arctan x}{2} - \frac{1}{2}(x - \arctan x)}_{F(x)} + C_1$$

$$\therefore \int_0^1 x \arctan x \, dx = F(1) - F(0) =$$

$$= \left[\frac{x^2 \arctan x}{2} - \frac{x}{2} + \frac{\arctan x}{2} \right]_0^1$$

$$= \frac{1^2 \cdot \arctan(1)}{2} - \frac{1}{2} + \frac{\arctan(1)}{2} - \frac{0}{2} - \frac{0}{2} + \frac{0}{2}$$

$$= \left[\arctan(1) = \pi/4 \iff \tan \pi/4 = 1 \right] =$$

$$= \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2} = \frac{1}{4}(\pi - 2)$$

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Ex Beräkna arean av området
$D = \{(x,y) : 1 \le x \le 4,\ y = e^{-\sqrt{x}}\}$

Lösning vi söker $(e^{-\sqrt{x}} > 0)$

$$\int_1^4 e^{-\sqrt{x}}\,dx = \int_1^4 f(x)\,dx = F(4) - F(1)$$

då $F'(x) = f(x)$

$$\int e^{-\sqrt{x}}\,dx = \left[\begin{array}{l} t = \sqrt{x} \Rightarrow t^2 = x \\ \Leftrightarrow x = t^2 \Rightarrow \\ \frac{dx}{dt} = 2t \Leftrightarrow dx = 2t\,dt \end{array}\right]$$

$$= \int e^{-t} \cdot 2t\,dt = 2\int e^{-t}\,t\,dt =$$

$$= \left[\int u'v\,dx = uv - \int uv'\,dx\right]$$

$$= \left[\begin{array}{l} u' = e^{-t},\ v = t \\ u = -e^{-t},\ v' = 1 \end{array}\right] =$$

$$= 2\left\{-e^{-t} \cdot t - \int -e^{-t}\underbrace{\frac{d}{dt}(t)}_{=1}\,dt\right\}$$

$$= 2\left\{-e^{-t}t - \underbrace{\int -e^{-t}\,dt}_{e^{-t}+C_1}\right\} =$$

$$= 2\left\{-e^{-t}t - e^{-t} + C_1\right\}$$

$$\therefore \int e^{-t}\,2t\,dt = -2e^{-t}(1+t) + C_1$$

$$\int e^{-\sqrt{x}}\,dx = \left[t = \sqrt{x}\right] = \underbrace{-2e^{-\sqrt{x}}(1+\sqrt{x})}_{F(x)} + C$$

$$\int_1^4 e^{-\sqrt{x}}\,dx = F(4) - F(1) =$$

$$= -2e^{-\sqrt{4}}(1+\sqrt{4}) - \left(-2e^{-\sqrt{1}}(1+\sqrt{1})\right)$$

$$= -2e^{-2}(1+2) - \left(-2e^{-1}(1+1)\right)$$

$$= -6e^{-2} + 4e^{-1}$$

$$= 4e^{-1} - 6e^{-2}$$

SVAR $\int_1^4 e^{-\sqrt{x}}\,dx = 4e^{-1} - 6e^{-2}$

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