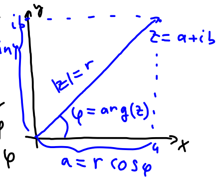
Förel 17. Komplexatal — Polynom- algebraiska ekvationer

Komplexa tal på polär form



$z = a+ib = r(\cos\varphi + i\sin\varphi)$

Def. De moivres formler

$$\begin{cases} \cos\varphi + i\sin\varphi = e^{i\varphi} \\ \cos\varphi - i\sin\varphi = e^{-i\varphi} \end{cases}$$

$$\cos\varphi = \frac{e^{i\varphi}+e^{-i\varphi}}{2}, \quad \sin\varphi = \frac{e^{i\varphi}-e^{-i\varphi}}{2i}$$

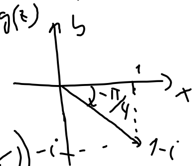
Varje $z \in \mathbb{C}$: $z = a+ib = |z|(\cos\varphi + i\sin\varphi)$
$= |z|e^{i\varphi}$

obs! $|e^{\pm i\varphi}| = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1$

$$e^{i\varphi}e^{i\beta} = e^{i(\varphi+\beta)}$$

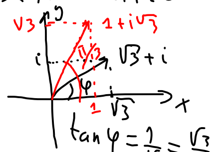
$$\frac{e^{i\varphi}}{e^{+i\beta}} = e^{i(\varphi-\beta)}$$

Ex1 skriv $z = 1-i$ på polär form



$z = 1-i = |z|e^{i\arg(z)}$
$= \sqrt{2}\,e^{i(-\pi/4)}$
$= \sqrt{2}\,e^{-i\pi/2} =$
$= \sqrt{2}\left(\cos\pi/4 - i\sin\pi/4\right)$

Ex2 skriv $\dfrac{(\sqrt{3}+i)^5}{1+i\sqrt{3}}$ på formen $a+ib$



$\tan\varphi = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
$\varphi = \pi/6$

$$\frac{(\sqrt{3}+i)^5}{1+i\sqrt{3}} = \frac{z_1^5}{z_2}$$

$z_1^5 = \left(|z_1|e^{i\varphi}\right)^5$
$= |z_1|^5 e^{5i\varphi}$
$= 2^5 e^{\frac{5\pi}{6}i}$

$z_2 = |z_2|e^{i\theta} = 2e^{i\pi/3}$

$$\frac{(\sqrt{3}+i)^5}{1+i\sqrt{3}} = \frac{2^{\cancel{5}\,4} e^{\frac{5\pi}{6}i - i\frac{\pi 2}{6}}}{\cancel{2}\,\cancel{e^{i\pi/3}}} \quad = \frac{3\pi i}{6} = \frac{\pi i}{2}$$

$= 2^4 e^{\frac{\pi i}{2}} = 16\left(\underbrace{\cos\frac{\pi}{2}}_{=0} + i\,\sin\pi/2\right)$ $=1$

$= 16i$

SVAR $\boxed{(\sqrt{3}+i)^5/(1+i\sqrt{3}) = 16i}$

Binomiska ekv: $z^n = w$, $n = 0, 1, 2, \dots$

Vi söker alla n-rötter till $z^n - w = 0$

Lösning steg 1 skriv ekv. på polärform.

$z = |z| e^{i\theta}$, $\theta = \arg(z)$, $w = |w| \cdot e^{i\varphi}$, $\varphi = \arg(w)$

$$z^n = w \iff \left(|z| e^{i\theta}\right)^n = |w| e^{i\varphi}$$

$$\iff |z|^n e^{in\theta} = |w| e^{i\varphi}$$

steg 2 lös ekv $|z|^n e^{in\theta} = |w| e^{i\varphi}$

$$\implies \begin{cases} |z|^n = |w| \\ n\theta = \varphi + 2k\pi \end{cases}$$

$\implies$ ty $\cos(\varphi + 2k\pi) = \cos\varphi$

$\sin(\varphi + 2k\pi) = \sin\varphi$

period $2k\pi$

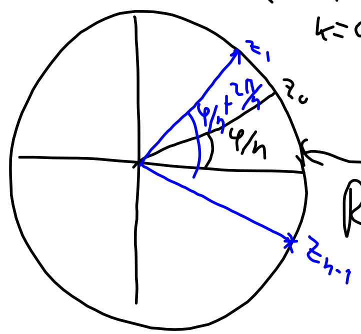
$$\implies \begin{cases} |z| = |w|^{1/n} \\ \theta = \frac{\varphi}{n} + \frac{2k}{n}\pi \end{cases}$$

$\underbrace{k = 0, 1, 2 \dots n-1}_{n \text{ rötter}}$

Svar $z^n = w$ har

n nollställen $z_k = |w|^{1/n} e^{i\left(\frac{\varphi}{n} + \frac{2k\pi}{n}\right)}$

$k = 0, 1, 2, \dots, n-1$



cirkel med Radie $= |w|^{1/n}$

Andragradsekvation med komplexa koefficienter

EXA Lös $(1+i)z^2+(2i-2)z+6-2i=0$

Lösning Steg 1 Gör koeff till z^2 blir 1

Då fås $z^2+\frac{2i-2}{1+i}z+\frac{6-2i}{1+i}=0 \quad (1)$

Förenkla vi

$$\frac{c+id}{a+ib}=\frac{(c+id)(a-ib)}{(a+ib)(a-ib)}=\frac{(c+id)(a-ib)}{a^2+b^2}=A+iB$$

$$\frac{2i-2}{1+i}=\frac{2(i-1)(1-i)}{(1+i)(1-i)}=\frac{\cancel{2}(i-1)(1-i)}{\cancel{2}}=i$$

$$\frac{6-2i}{1+i}=\frac{\cancel{2}(3-i)(1-i)}{\underbrace{(1+i)(1-i)}_{\cancel{2}}}=(3-i)(1-i)=2-4i$$

$$(1)\Rightarrow \boxed{z^2+2iz+2-4i=0 \quad (2)}$$

Steg 2 Kvadrat komplettera

$$(A+B)^2=A^2+2AB+B^2$$

$$\Longleftrightarrow A^2+2AB=(A+B)^2-B^2$$

$$\underbrace{z^2+2iz}_{(z+i)^2-\underbrace{i^2}_{=-1}}+2-4i=0 \Longleftrightarrow$$

$$(z+i)^2+1+2-4i=0$$

$$\Longleftrightarrow \boxed{(z+i)^2=-3+4i} \quad (3)$$

Steg 3 Lös (3) via substitution.

Sätt $z+i=x+iy$

$$(3)\Rightarrow (x+iy)^2=-3+4i$$

$$\Rightarrow x^2-y^2+i2xy=-3+4i$$

$$\Rightarrow \begin{cases} x^2-y^2=-3 & (Ek1) \\ \underbrace{\cancel{2}xy}=\cancel{4}2 & (Ek2) \end{cases}$$

Säger att x och y har samma tecken

Lös t.ex. ur (ek2) $y=2/x$ in i ekv (1)

$$\Rightarrow x^2-\frac{4}{x^2}=-3 \Longleftrightarrow x^4-4=-3x^2$$

$$\Longleftrightarrow x^4+3x^2-4=0 \quad (4)$$

Lös ekv (4) via t.ex. $x^2=t\geq 0$

$$\Rightarrow t^2+3t-4=0 \Rightarrow \cancel{t=-4},\ t=1$$

$$t=1=x^2 \Rightarrow x=\pm 1 \quad \text{Bra}$$

Men $y=\frac{2}{x} \Rightarrow y=\pm 2$

Steg 4 Vet att $z+i=x+iy$

$$z=x+iy-i$$

$x=1, y=2 \Rightarrow z=1+i$
$x=-1, y=-2 \Rightarrow z=-1-3i$] SVARET!

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Polynom – algebraiska ekvationer

Varje polynom $p_n(z) = \sum_{k=0}^{n} a_k z^k$

$= a_0 + a_1 z + \dots + a_n z^n$ där $a_n \neq 0$

av grad n har n-nollställen

om vi kallar dessa $z_1, z_2, \dots, z_n$

$P_n(z) = \text{konstant } (z-z_1)(z-z_2)\cdots(z-z_n)$

$= \text{konstant } \prod_{k=1}^{n} (z-z_k)$

(Faktorisering)

<u>EXB</u> Faktorisera $2z^3 - 2(1+2i)z^2 + 6(-1+i)z + 4i(1-i)$

Som har ett nollställe $z=2$

<u>Lösning</u>

$2z^3 - 2(1+2i)z^2 + 6(-1+i)z + 4i(1-i) =$

$(z-2)(az^2+bz+c)$ (1)

HL(1): $az^3 + (b-2a)z^2 + (c-2b)z - 2c = VL(1)$

$\Rightarrow \begin{cases} 2 = a \\ -2(1+2i) = (b-2a) \\ 6(-1+i) = c-2b \\ 4i(1-i) = -2c \end{cases} \Rightarrow \begin{matrix} a = 2 \\ b = 2-4i \\ c = -2-2i \end{matrix}$

HL(1): $(z-2)\underbrace{\left(2z^2 + (2-4i)z - 2-2i\right)}_{\text{Löses via EXA}} = \text{c}$

<u>SVAR</u> Den givna ekv:

$(z-2)(z-i)(z+1-i)$

$z_1 = 2,\ z_2 = i,\ z_3 = -1+i$

polynom med reella koefficienter

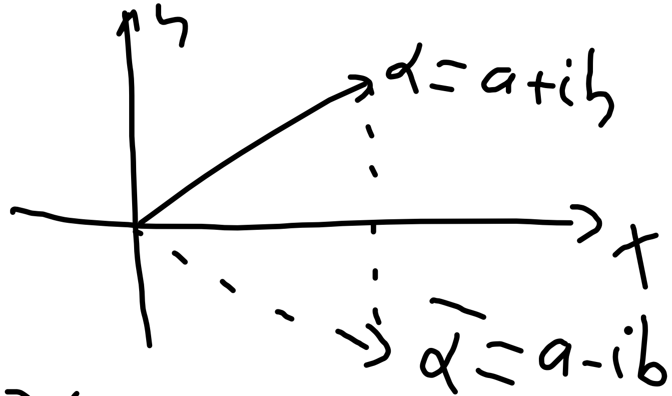
$$P_n(t) = a_0 + a_1 t + \dots + a_n t^n,$$

$a_0, a_1, \dots, a_n$ Reella

om $A \in \mathbb{R} \iff A = \bar{A}$

Om $\alpha = a+ib$ är en rot till $P_n(t)=0$

$\Rightarrow \bar{\alpha} = a-ib$ är också en rot.



Ty om $P_n(\alpha) = a_0 + a_1\alpha + \dots + a_n\alpha^n = 0$

$$\Rightarrow \overline{P_n(\alpha)} = \overline{a_0} + \overline{a_1\alpha} + \dots + \overline{a_n\alpha^n} = 0$$

$$= a_0 + a_1\bar{\alpha} + \dots + a_n\bar{\alpha}^n = 0$$

$$\Rightarrow P_n(\bar{\alpha}) = 0$$

Ex C $\quad z^4 - 2z^3 - 2z^2 - 2z - 3 = 0 \quad (*)$

har en rot $z_1 = i$

Bestäm samtliga rötter (tenta tal)

Lösning $(*)$ har reella koefficienten

$\Rightarrow$ om $z_1 = i$ är en rot till $(*)$

$\Rightarrow z_2 = \bar{z}_1 = \bar{i} = -i$ är också en rot.

$(*)$ kan skrivas (faktoriseras)

1. $z^4 - 2z^3 - 2z^2 - 2z - 3 =$

$$= \underbrace{(z-i)(z+i)}_{\underbrace{z^2 + iz - iz + 1}_{z^2+1}}(1z^2 + bz + c)$$

$$= (z^2+1)(z^2+bz+c)$$

$$= z^4 + bz^3 + cz^2 + z^2 + bz + c$$

$$= z^4 + \underset{\uparrow -2}{b}z^3 + \underbrace{(c+1)}_{-2}z^2 + \underbrace{b}_{-2}z + \underbrace{c}_{-3}$$

$$= (z^2+1)(z^2 - 2z - 3)$$

Vi löser då $\underbrace{(z^2+1)}_{z = \pm i}\underbrace{(z^2 - 2z - 3)}_{z = -1,\, 3} = 0$

SVAR Rötterna till den givna ekvationen är $\pm i, -1, 3$

Räknelagar med komplexa tal

1. $z_1 = z_2 \iff \begin{cases} Re[z_1] = Re[z_2] \\ Im[z_1] = Im[z_2] \end{cases}$

om $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

$$z_1 = z_2 \iff \begin{cases} r_1 = r_2 \\ \theta_1 = \theta_2 + 2n\pi, \; n = 0, \pm 1, \pm 2, \ldots \end{cases}$$

3. om $z = re^{i\theta}$

$$\Rightarrow z^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$$

4. om $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

5. $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$