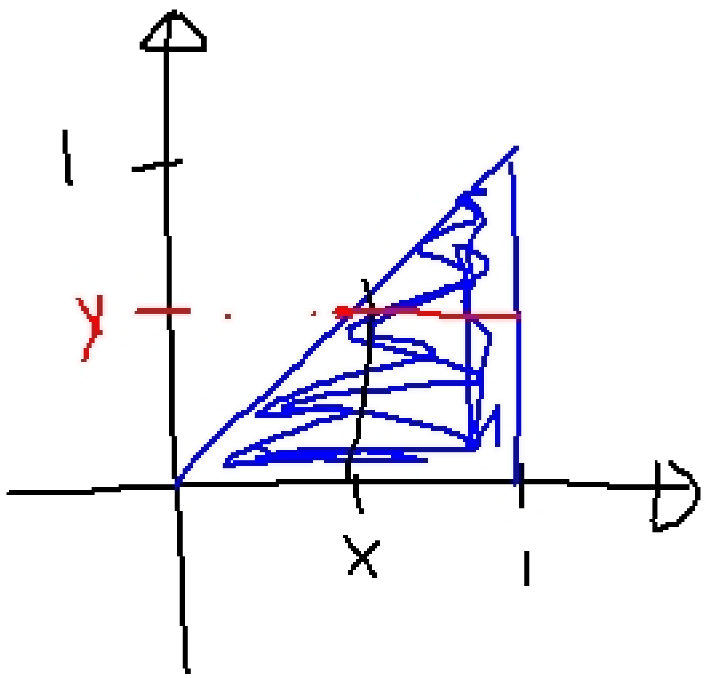
(Ex 9.7) Beräkna $\displaystyle\int_0^1 \left(\int_y^1 e^{x^2} dx \right) dy$

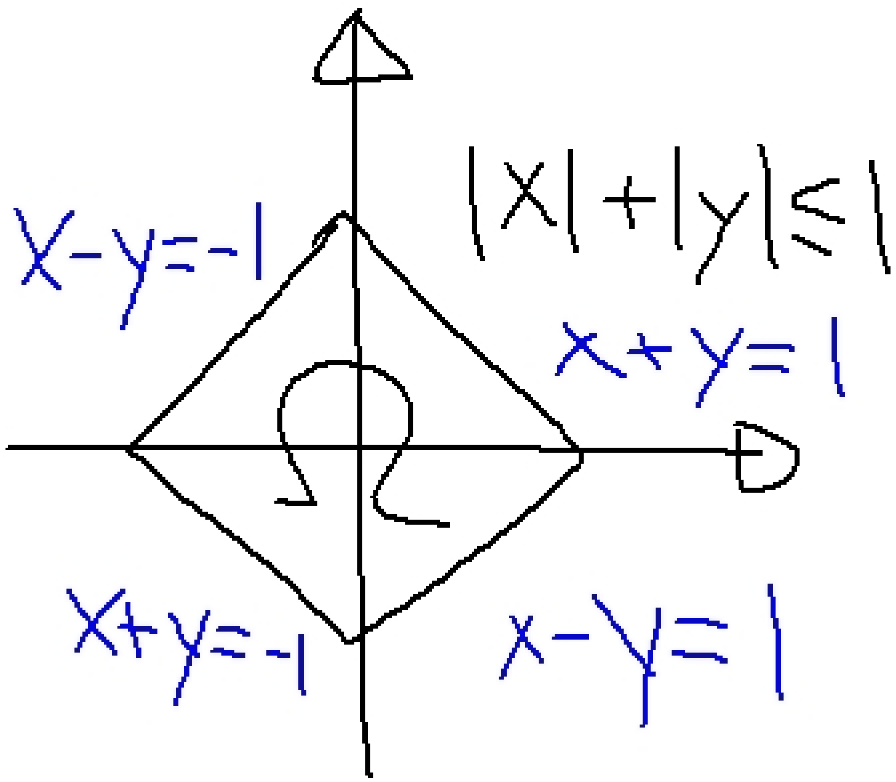


$$= \int_0^1 \left(\int_0^x e^{x^2} dy \right) dx =$$

$$= \int_0^1 \left[e^{x^2} y \right]_0^x dx = \int_0^1 e^{x^2} x \, dx =$$

$$= \left[\frac{e^{x^2}}{2} \right]_0^1 = \frac{e}{2} - \frac{1}{2}$$

2. Berikna $\iint_{\Omega} x^2-y^2\,dx\,dy$



$T^{-1}: \begin{cases} u = x+y \\ v = x-y \end{cases}$ $\quad T: \begin{cases} x = \frac{u+v}{2} \\ y = \frac{u-v}{2} \end{cases}$

$|\det T| = \frac{1}{2}$

$$\iint_{\Omega} x^2-y^2\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} uv\,\frac{1}{2}\,du\,dv =$$

$$= \int_{-1}^{1} \left[\frac{u^2 v}{4}\right]_{-1}^{1} dv = 0$$

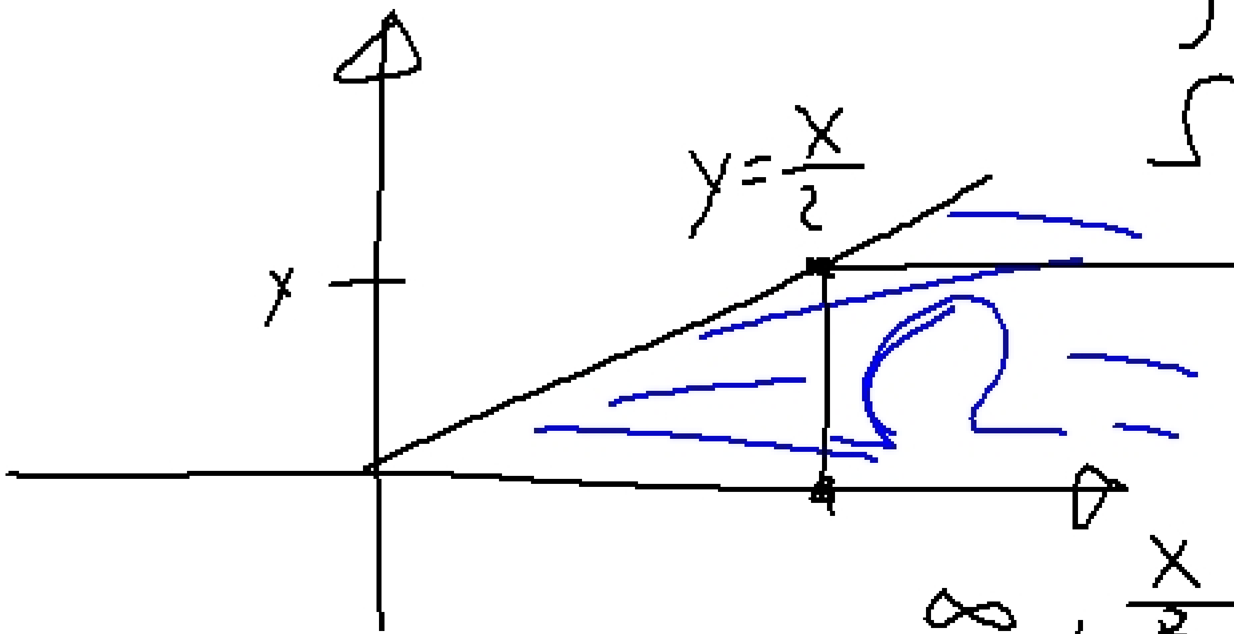
3. Beräkna $\displaystyle\iint_{2x^2+y^2\leq 1} xy\,dx\,dy$

$$T:\begin{cases} x = r\cos\varphi\frac{1}{\sqrt{2}} \\ y = r\sin\varphi \end{cases} \qquad \det\begin{pmatrix} \frac{\cos\varphi}{\sqrt{2}} & -\frac{r\sin\varphi}{\sqrt{2}} \\ \sin\varphi & r\cos\varphi \end{pmatrix} = \frac{r}{\sqrt{2}}$$

$$\int_0^{2\pi}\int_0^1 r\cos\varphi\frac{1}{\sqrt{2}}\,r\sin\varphi\,\frac{r}{\sqrt{2}}\,dr\,d\varphi = \int_0^{2\pi}\frac{\cos\varphi\sin\varphi}{8}\,d\varphi$$

$$= \int_0^{2\pi}\frac{\sin 2\varphi}{16}\,d\varphi = \left[-\frac{\cos 2\varphi}{32}\right]_0^{2\pi} = 0$$

4. Berikna $\iint\limits_{\Omega} e^{y-x}\,dx\,dy$



$x \geq 2y \geq 0$

$$\iint\limits_{\Omega} e^{y-x}\,dx\,dy = \int_0^\infty \left(\int_0^{\frac{x}{2}} e^{y-x}\,dy\right)dx =$$

$$= \int_0^\infty \left[e^{y-x}\right]_0^{\frac{x}{2}} dx = \int_0^\infty e^{-\frac{x}{2}} - e^{-x}\,dx$$

$$= \left[-2e^{-\frac{x}{2}} + e^{-x}\right]_0^\infty = 2-1 = 1$$

5. Beräkna $\displaystyle\iiint_K x \, dx\, dy\, dz$

då $K = \{(x,y,z);\ x^2+y^2+z^2 \leq 1\}$.

$$T: \begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \qquad |\det J_T| = r^2\sin\theta$$

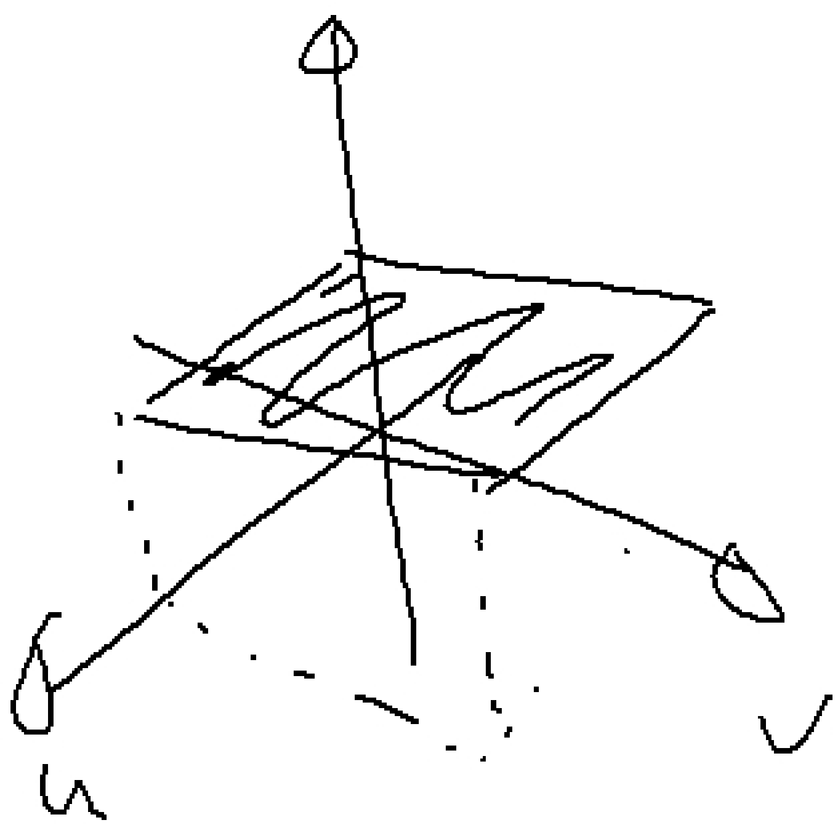
$$\int_0^{\pi}\left(\int_0^{1}\left(\int_0^{2\pi} r\sin\theta\cos\varphi\, r^2\sin\theta \, d\varphi\right) dr\right) d\theta = \int_0^{\pi}\left(\int_0^{1}\left[r^3\sin^2\theta\sin\varphi\right]_0^{2\pi} dr\right) d\theta$$

$$= 0$$

6. Beräkna arean av

$$\vec{r}(u,v) = (u, v, 1)$$

$$0 \le u \le 1, \quad 0 \le v \le 1.$$



$$A = \iint_{\Omega} \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du\,dv$$

$$= \int_0^1 \int_0^1 |(0,0,1)|\, du\,dv = \int_0^1 \int_0^1 1\, du\,dv$$

$$= 1$$

Samma uppgift fast sedd som en graf: $z = f(x,y) = 1$

Arean ges då av

$$\iint_{\substack{0\le x\le 1\\ 0\le y\le 1}} \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy = \int_0^1\int_0^1 dx\,dy = 1$$

7. Beräkna linjeintegralen

$$\oint_{\Gamma} \vec{F}(\vec{r}) \cdot d\vec{r}$$

då $\vec{F}(x,y) = (\overset{P}{-y}, \overset{Q}{x})$ och

Γ är enhetscirkeln $x^2+y^2=1$.

Lösn 1: Γ $(\cos t, \sin t)$,

$$\int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt = \int_0^{2\pi} dt = 2\pi$$

Lösn 2: Greens formel ger

$$\oint_{\Gamma} \vec{F}(\vec{r})\cdot d\vec{r} = \iint_{x^2+y^2\leq 1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\,dy$$

$$= \iint_{x^2+y^2\leq 1} 1-(-1)\, dx\,dy = 2\iint_{x^2+y^2\leq 1} dx\,dy = 2\pi$$