Om f inte är större än eller lika med noll och $\iint_\Omega f(x,y)\,dx\,dy$ divergerar så kan det hända att enkelintegralerna existerar:

Ex 9.15: $\iint_\Omega x e^{-x^2}\,dx\,dy$ där

dubbelintegralen är divergent för

$$\iint_\Omega \left|x e^{-x^2}\right| dx\,dy = 2\int_0^\infty \left(\int_0^\infty x e^{-x^2}\,dx\right) dy = 2\int_0^\infty \left[\frac{-e^{-x^2}}{2}\right]_0^\infty dy$$

$$= \int_0^\infty dy = \infty$$

men vi har

$$\int_0^\infty \left(\int_{-\infty}^\infty x e^{-x^2} \, dx \right) dy = \int_0^\infty \left[-\frac{e^{-x^2}}{2} \right]_{-\infty}^\infty dy =$$

$$= \int_0^\infty 0 \, dy = 0.$$

Ex/övn: Beräkna $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

Lösn: $\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)$

$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\right)dy = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy$

$= \iint_{\mathbb{R}^2} e^{-r^2}\,r\,dr\,d\varphi = \int_0^{2\pi}\left(\int_0^{\infty} re^{-r^2}\,dr\right)d\varphi = \frac{2\pi}{2} = \pi$

Trippelintegraler

Definieras som tidigare med Riemannsummor

$$S_i = \sum_{h,k,l} f(\xi_h, \eta_k, \zeta_l)\,\Delta x_h \Delta y_k \Delta z_l$$

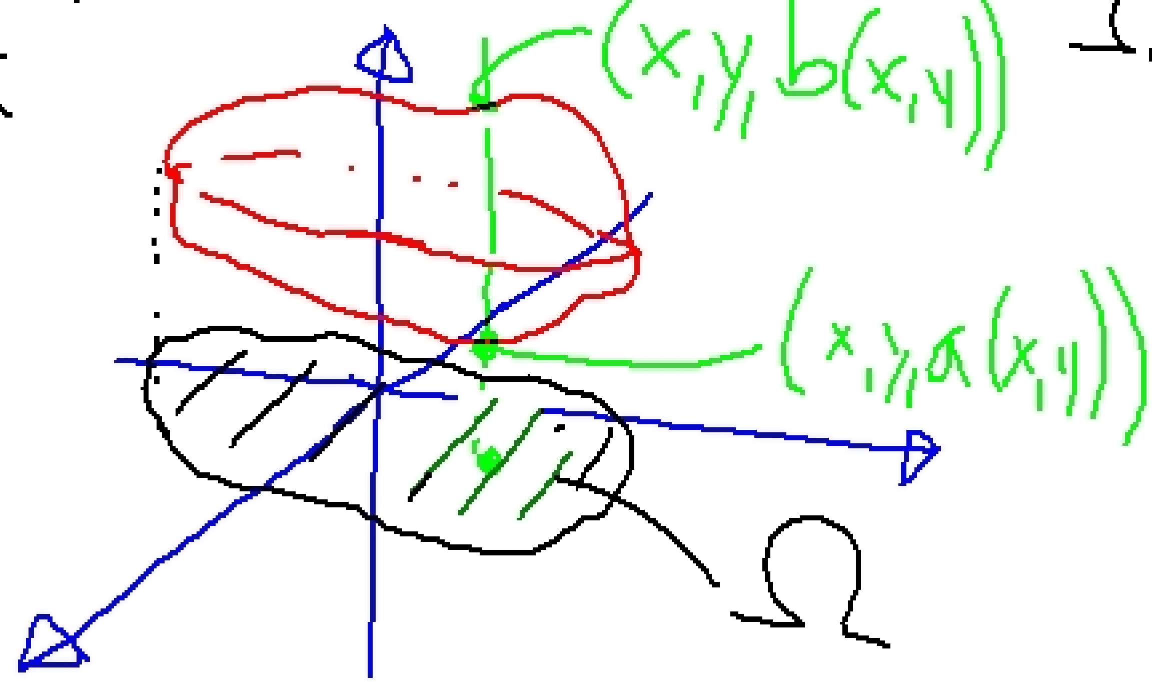
$$\iiint_K f(x,y,z)\,dx\,dy\,dz = \lim S_i$$

där gränsvärdet tas över allt finare indelningar.

Vi kan beräkna volymen hos kroppen K m h a trippelintegralen

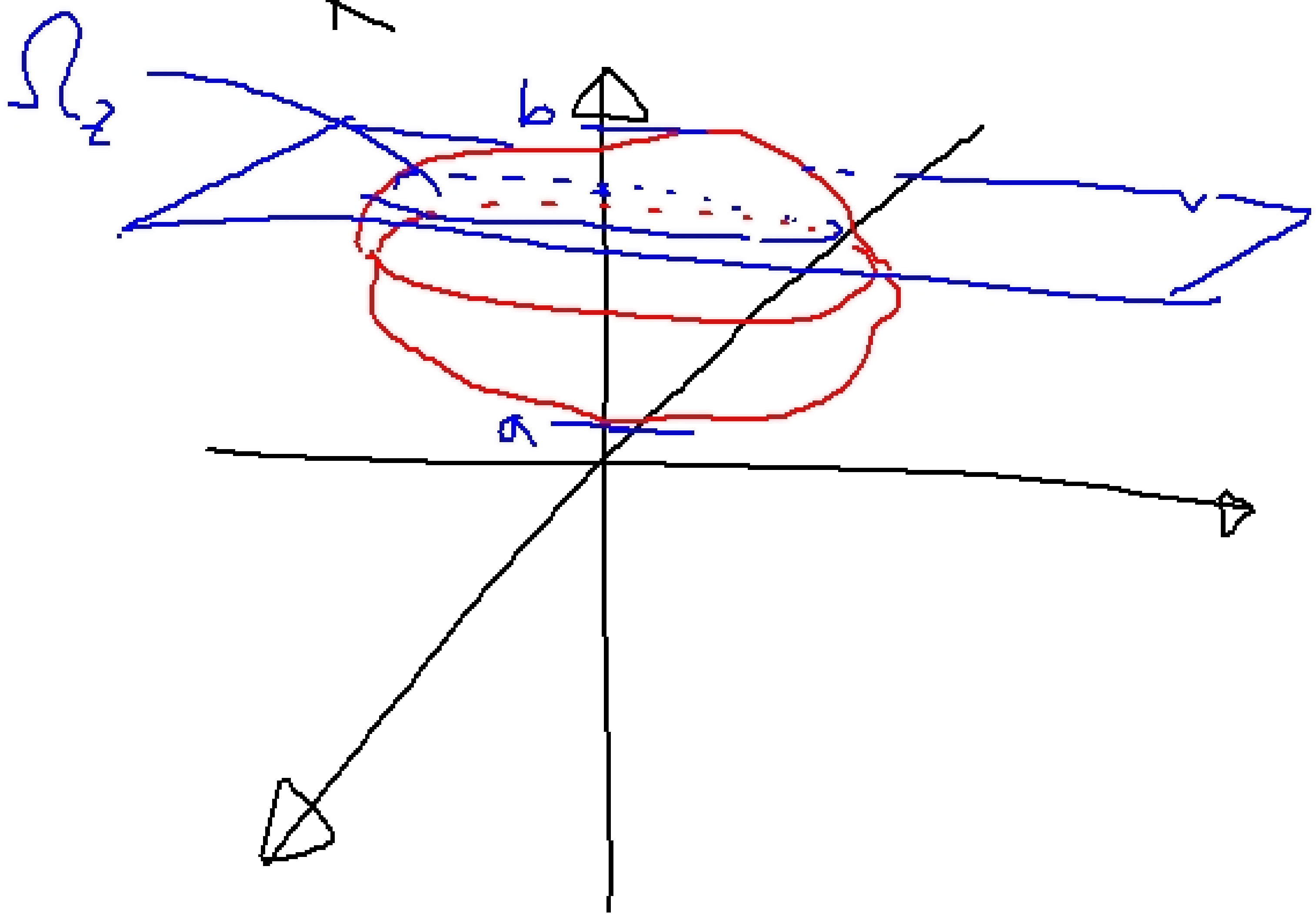
$$\iiint_K dx\,dy\,dz.$$

Sats: $$\iiint_K f(x,y,z)\,dx\,dy\,dz = \iint_\Omega \left(\int_{a(x,y)}^{b(x,y)} f(x,y,z)\,dz \right) dx\,dy$$



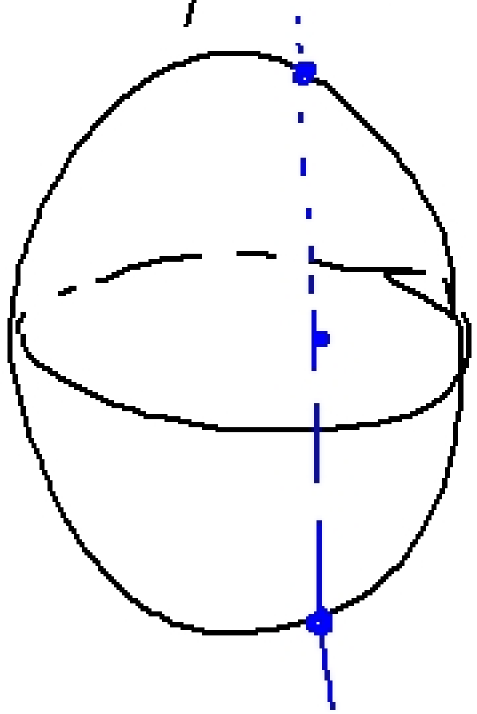
man kan också ta z-integralen ytterst

$$\iiint_K f(x,y,z)\,dx\,dy\,dz = \int_a^b \left(\iint_{\Omega_z} f(x,y,z)\,dx\,dy \right) dz$$



Ex/övn: Beräkna volymen hos enhetsklotet $x^2+y^2+z^2 \le 1$.

$$\iiint_{x^2+y^2+z^2\le 1} dx\,dy\,dz = \iint_{x^2+y^2\le 1} \left(\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz \right) dx\,dy$$



$$= \iint_{x^2+y^2\le 1} 2\sqrt{1-x^2-y^2}\, dx\,dy =$$

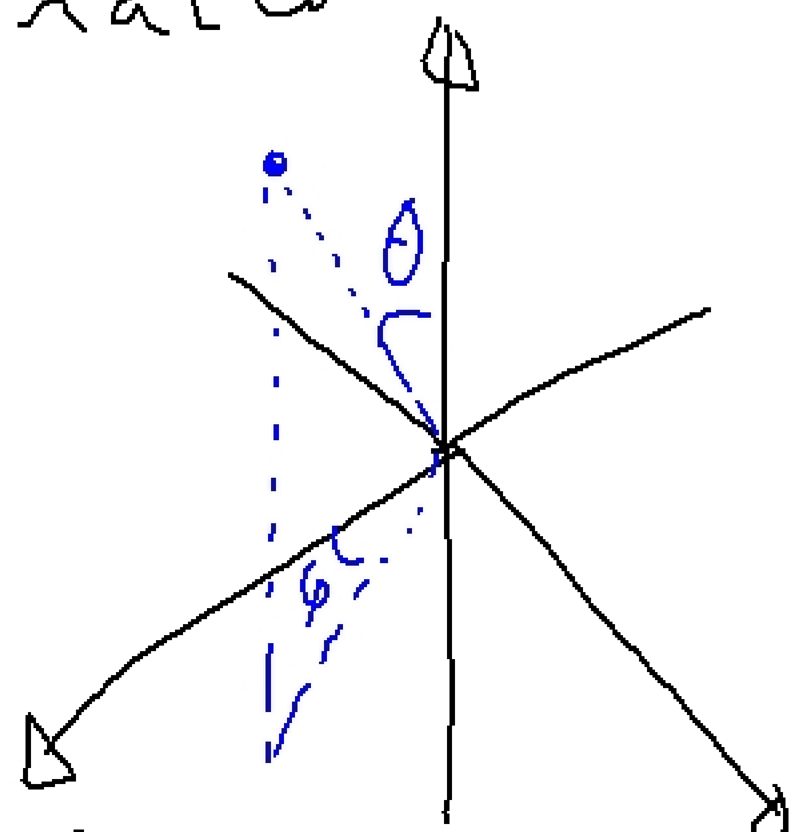
$$= \int_0^{2\pi} \left(\int_0^1 2\sqrt{1-r^2}\, r\,dr \right) d\varphi =$$

$$= 2\pi \int_0^1 2\sqrt{1-r^2}\, r\,dr = 2\pi \left[(1-r^2)^{\frac{3}{2}} \left(-\frac{2}{3}\right)\right]_0^1 =$$

$$= \frac{4\pi}{3}.$$

Sfäriska koordinater

$$T: \begin{cases} x = r\cos\varphi \sin\theta \\ y = r\sin\varphi \sin\theta \\ z = r\cos\theta \end{cases}$$



$$\left|\det J_T(r,\varphi,\theta)\right| = r^2 \sin\theta$$

Sats:

$$\iiint_K f(x,y,z)\,dx\,dy\,dz =$$

$$= \iiint_{K'} f(x(u,v,w), y(u,v,w), z(u,v,w))\,|\det J_T(u,v,w)|\,du\,dv\,dw$$

där T tar K' till K.

Ex/övn: Beräkna volymen hos enhets-klotet mha sfäriska koordinater.

$$\iiint_{K'} |\det J_T(r,\varphi,\theta)| \, dr \, d\varphi \, d\theta =$$

$$= \int_0^{\pi} \left(\int_0^{2\pi} \left(\int_0^1 r^2 \sin\theta \, dr \right) d\varphi \right) d\theta =$$

$$= \int_0^{\pi} \left(\int_0^{2\pi} \left[\frac{r^3}{3} \sin\theta \right]_0^1 d\varphi \right) d\theta = \int_0^{\pi} \frac{2\pi}{3} \sin\theta \, d\theta =$$

$$= \left[\frac{2\pi}{3} (-\cos\theta) \right]_0^{\pi} = \frac{2\pi}{3} + \frac{2\pi}{3} = \frac{4\pi}{3}$$