Några lösta tal om Nablaräkning 11.5abc,11.6ab,11.10abc,11.11abc

11.5a. Låt a och b vara reella konstanter samt $\boldsymbol{F}$ och $\boldsymbol{G}$ två differentierbara vektorfält. Verifera att

$$\operatorname{div}(a\boldsymbol{F}+b\boldsymbol{G})=a\operatorname{div}\boldsymbol{F}+b\operatorname{div}\boldsymbol{G}$$

Lösning:

För en funktion $\boldsymbol{Y}:\boldsymbol{R}^n\to\boldsymbol{R}^n$ har vi att

$$\operatorname{div}\boldsymbol{Y}=\sum_{i=1}^{n}\frac{\partial Y_i}{\partial x_i}.$$

Vi får alltså

$$\operatorname{div}(a\boldsymbol{F}+b\boldsymbol{G})=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(aF_i+bG_i\right)=\lceil\text{Derivation av summa}\rceil=$$

$$=\sum_{i=1}^{n}\left(\frac{\partial}{\partial x_i}aF_i+\frac{\partial}{\partial x_i}bG_i\right)=\lceil a,b\text{ konstanter}\rceil=\sum_{i=1}^{n}\left(a\frac{\partial F_i}{\partial x_i}+b\frac{\partial G_i}{\partial x_i}\right)=$$

$$=\sum_{i=1}^{n}a\frac{\partial F_i}{\partial x_i}+\sum_{i=1}^{n}b\frac{\partial G_i}{\partial x_i}=a\sum_{i=1}^{n}\frac{\partial F_i}{\partial x_i}+b\sum_{i=1}^{n}\frac{\partial G_i}{\partial x_i}=a\operatorname{div}\boldsymbol{F}+b\operatorname{div}\boldsymbol{G}$$

Vilket skulle verifieras.

11.5b. Låt u och $\boldsymbol{F}$ vara differentierbara funktioner av typ $\boldsymbol{R}^3\to\boldsymbol{R}$ respektive $\boldsymbol{R}^3\to\boldsymbol{R}^3$. Verifiera att

$$\operatorname{div}(u\boldsymbol{F})=(\operatorname{grad}u)\cdot\boldsymbol{F}+u\operatorname{div}\boldsymbol{F}$$

Lösning:

Vi får

$$\operatorname{div}(u\boldsymbol{F})=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot\left(uF_1,uF_2,uF_3\right)=$$

$$=\frac{\partial}{\partial x}\left(uF_1\right)+\frac{\partial}{\partial y}\left(uF_2\right)+\frac{\partial}{\partial z}\left(uF_3\right)=\lceil\text{Produktregeln}\rceil=$$

$$=\frac{\partial u}{\partial x}F_1+u\frac{\partial F_1}{\partial x}+\frac{\partial u}{\partial y}F_2+u\frac{\partial F_2}{\partial y}+\frac{\partial u}{\partial z}F_3+u\frac{\partial F_3}{\partial z}=$$

$$=\left(F_1,F_2,F_3\right)\cdot\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)+u\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)=$$

$$=(\operatorname{grad}u)\cdot\boldsymbol{F}+u\operatorname{div}\boldsymbol{F}$$

Vilket skulle verifieras.

11.5c. Låt u vara en $\boldsymbol{R}^3\to\boldsymbol{R}$ funktion med kontinuerliga andraderivator. Verifiera att

$$\operatorname{div}(\operatorname{grad}u)=\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}$$

Lösning:

Vi får

$$\operatorname{div}(\operatorname{grad} u)=\operatorname{div}\left(\frac{\partial \mathrm{u}}{\partial \mathrm{x}},\frac{\partial \mathrm{u}}{\partial \mathrm{y}},\frac{\partial \mathrm{u}}{\partial \mathrm{z}}\right)=\left(\frac{\partial}{\partial \mathrm{x}},\frac{\partial}{\partial \mathrm{y}},\frac{\partial}{\partial \mathrm{z}}\right)\cdot\left(\frac{\partial \mathrm{u}}{\partial \mathrm{x}},\frac{\partial \mathrm{u}}{\partial \mathrm{y}},\frac{\partial \mathrm{u}}{\partial \mathrm{z}}\right)=$$

$$=\frac{\partial}{\partial \mathrm{x}}\frac{\partial \mathrm{u}}{\partial \mathrm{x}}+\frac{\partial}{\partial \mathrm{y}}\frac{\partial \mathrm{u}}{\partial \mathrm{y}}+\frac{\partial}{\partial \mathrm{y}}\frac{\partial \mathrm{u}}{\partial \mathrm{y}}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$

Vilket skulle verifieras.

Anmärkning I vektoranalysen använder man beteckningen ∇ för vektorn $\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$, dvs $\nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$. Relationen ovan kan med denna notation skrivas

$$\nabla\cdot\nabla u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$

och ibland tillåter man sig att skriva

$$\nabla\cdot\nabla u=\nabla^2 u\,.$$

Vidare har operatorn $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ fått namnet *Laplace*-operatorn och beteckningen Δ, dvs $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$. Med notationen enligt ovan kan man alltså skriva

$$\Delta=\nabla^2\,.$$

Se även boken sidan 299.

11.6a. Beräkna

$$\operatorname{div}(|\boldsymbol{r}|\cdot\boldsymbol{v})$$

där **v** är en konstant vektor.

Lösning:

$$\operatorname{div}(|\mathbf{r}|\cdot\mathbf{v})=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot(|\mathbf{r}|\cdot\mathbf{v})=\lceil\text{enligt övning 11.5b}\rceil=$$

$$=(\operatorname{grad}|\mathbf{r}|)\cdot\mathbf{v}+|\mathbf{r}|\underbrace{\operatorname{div}\mathbf{v}}_{=0}=\left(\operatorname{grad}\left(\sqrt{\mathrm{x}^2+y^2+z^2}\right)\right)\cdot\mathbf{v}=$$

$$=\left(\frac{x}{\sqrt{\mathrm{x}^2+y^2+z^2}},\frac{y}{\sqrt{\mathrm{x}^2+y^2+z^2}},\frac{z}{\sqrt{\mathrm{x}^2+y^2+z^2}}\right)\cdot\mathbf{v}=$$

$$=\frac{\mathbf{r}\cdot\mathbf{v}}{|\mathbf{r}|}$$

SVAR: $\frac{\mathbf{r}\cdot\mathbf{v}}{|\mathbf{r}|}$

11.6b. Beräkna

$$\operatorname{div}(|\boldsymbol{r}|\cdot\boldsymbol{r})$$

där **v** är en konstant vektor.

Lösning:

$$\operatorname{div}(|\mathbf{r}|\cdot\mathbf{r}) = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot(|\mathbf{r}|\cdot\mathbf{r}) = \lceil \text{enligt övning 11.5b} \rceil =$$

$$= (\operatorname{grad}|\mathbf{r}|)\cdot\mathbf{r} + |\mathbf{r}|\underbrace{\operatorname{div}\mathbf{r}}_{=3} =$$

$$= \left(\frac{x}{\sqrt{x^2+y^2+z^2}},\frac{y}{\sqrt{x^2+y^2+z^2}},\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\cdot\mathbf{r} + 3|\mathbf{r}| =$$

$$= \frac{\mathbf{r}\cdot\mathbf{r}}{|\mathbf{r}|} + 3|\mathbf{r}| = \frac{\mathbf{r}^2}{\sqrt{\mathbf{r}^2}} + 3|\mathbf{r}| = \sqrt{\frac{\mathbf{r}^4}{\mathbf{r}^2}} + 3|\mathbf{r}| = |\mathbf{r}| + 3|\mathbf{r}| = 4|\mathbf{r}|$$

SVAR: $4|\boldsymbol{r}|$

11.10a. Låt a och b vara reella konstanter samt $\boldsymbol{F}$ och $\boldsymbol{G}$ två differentierbara vektorfält. Verifera att

$$\operatorname{rot}(a\boldsymbol{F} + b\boldsymbol{G}) = a\operatorname{rot}\boldsymbol{F} + b\operatorname{rot}\boldsymbol{G}$$

Lösning:

Allmänt har vi att $\operatorname{rot}\mathbf{Y} = \nabla\times\mathbf{Y} = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\times\mathbf{Y}$.

Här får vi att

$$\operatorname{rot}(a\mathbf{F} + b\mathbf{G}) = \nabla\times(a\mathbf{F} + b\mathbf{G}) = \nabla\times(a\mathbf{F}) + \nabla\times(b\mathbf{G}) =$$

$$= \lceil a,b \text{ konstanter} \rceil = a\nabla\times\mathbf{F} + b\nabla\times\mathbf{G} =$$

$$= a\operatorname{rot}\mathbf{F} + b\operatorname{rot}\mathbf{G}$$

vilket skulle veriferas.

11.10b. Låt u och $\boldsymbol{F}$ vara differentierbara funktioner av typ $\boldsymbol{R}^3 \to \boldsymbol{R}$ respektive $\boldsymbol{R}^3 \to \boldsymbol{R}^3$. Verifiera att

$$\operatorname{rot}(u\boldsymbol{F}) = (\operatorname{grad} u)\times\boldsymbol{F} + u\cdot(\operatorname{rot}\boldsymbol{F})$$

Lösning:

Vi får

$$\operatorname{rot}(u\mathbf{F}) = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\times(uF_1,uF_2,uF_3) =$$

$$= \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ uF_1 & uF_2 & uF_3 \end{vmatrix} = \hat{\mathbf{e}}_x\left(\frac{\partial}{dy}(uF_3) - \frac{\partial}{dz}(uF_2)\right) - \hat{\mathbf{e}}_y\left(\frac{\partial}{dx}(uF_3) - \frac{\partial}{dz}(uF_1)\right) +$$

$$+\hat{\mathbf{e}}_z\left(\frac{\partial}{dx}(uF_2) - \frac{\partial}{dy}(uF_1)\right) = \hat{\mathbf{e}}_x\left(u'_yF_3 + u\frac{\partial F_3}{\partial y} - u'_zF_2 - u\frac{\partial F_2}{\partial z}\right) -$$

$$-\hat{\mathbf{e}}_y\left(u'_xF_3 + u\frac{\partial F_3}{\partial x} - u'_zF_1 - u\frac{\partial F_1}{\partial z}\right) + \hat{\mathbf{e}}_z\left(u'_xF_2 + u\frac{\partial F_2}{\partial x} - u'_yF_1 - u\frac{\partial F_1}{\partial y}\right) =$$

$$= \hat{\mathbf{e}}_x\left(u'_yF_3 - u'_zF_2 + u\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\right) - \hat{\mathbf{e}}_y\left(u'_xF_3 - u'_zF_1 + u\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\right) +$$

$$+\hat{\mathbf{e}}_z\left(u'_xF_2-u'_yF_1+u\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\right)=$$

$$=\begin{vmatrix}\hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z\\ u'_x & u'_y & u'_z\\ F_1 & F_2 & F_3\end{vmatrix}+u\begin{vmatrix}\hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}=(\text{grad }u)\times\mathbf{F}+u\text{rot}\mathbf{F}$$

Vilket skulle verifieras.

<u>Anmärkning</u> Som synes blir räkningarna ganska omständiga. Som allternativ kan man använda sk formell nablaräkning. Formel nablaräkning går ut på att man delar upp det ursprungliga uttrycket i en summa och sedan låter nabla-operatorn verka på ett element i varje summa. För detaljer se boken kapitel 11.5 och boken *Vektoranalys* av Anders Ramgaard.

Här får vi

$$\text{rot}(u\boldsymbol{F})=\nabla\times(u\boldsymbol{F})=\lceil\text{Formell nablaräkning}\rceil=\nabla\times((u^*)\boldsymbol{F})+\nabla\times(u(\boldsymbol{F}^*))=$$
$$=(\nabla(u^*))\times\boldsymbol{F}+u(\nabla\times(\boldsymbol{F}^*))=(\nabla u)\times\boldsymbol{F}+u(\nabla\times\boldsymbol{F})=$$
$$=(\text{grad u})\times\boldsymbol{F}+u\text{rot}\boldsymbol{F}$$

Här har de faktorer som nablaoperatorn skall verka på markerats med (*).

11.10c. Låt u och $\boldsymbol{F}$ vara differentierbara funktioner med kontinuerliga andraderivator definierade på $\mathbf{R}^3$. Förenkla

$$\text{rot}(\text{grad}u)\quad (1)$$

och

$$\text{div}(\text{rot }u).\quad (2)$$

Lösning:

Vi börjar med (2). Komponenträkning ger

$$\text{div}(\text{rot }\mathbf{F})=\nabla\cdot(\nabla\times\mathbf{F})=\begin{vmatrix}\dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}=\frac{\partial^2F_3}{\partial x\partial y}-\frac{\partial^2F_2}{\partial x\partial z}-\frac{\partial^2F_3}{\partial y\partial x}+\frac{\partial^2F_1}{\partial y\partial z}+\frac{\partial^2F_2}{\partial z\partial x}-\frac{\partial^2F_1}{\partial z\partial y}=0$$

.

Vidare får vi (1) till

$$\text{rot}(\text{grad }u)=\nabla\times(\nabla u)=\begin{vmatrix}\hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ u'_x & u'_y & u'_z\end{vmatrix}=\hat{\mathbf{e}}_x\left(u''_{zy}-u''_{yz}\right)-\hat{\mathbf{e}}_y\left(u''_{zx}-u''_{xz}\right)+$$

$$+\hat{\mathbf{e}}_z\left(u''_{yx}-u''_{xy}\right)=\mathbf{0}$$

ty u har kontinuerliga andraderivator.

SVAR: (1)=**0**,(2)=0.

11.11a. Beräkna

$$\operatorname{rot}\left(|\mathbf{r}|\,\mathbf{v}\right)$$

där **v** är en konstant vektor.

Lösning:

Resultatet från övning 11.10b ger att

$$\operatorname{rot}\left(|\mathbf{r}|\,\mathbf{v}\right)=\left(\operatorname{grad}|\mathbf{r}|\right)\times\mathbf{v}+|\mathbf{r}|\,\mathrm{r}\underbrace{\mathrm{ot}\mathbf{v}}_{=0}=\left(\operatorname{grad}\sqrt{\mathrm{x}^2+y^2+z^2}\right)\times\mathbf{v}=$$

$$=\left(\frac{x}{\sqrt{\mathrm{x}^2+y^2+z^2}},\frac{y}{\sqrt{\mathrm{x}^2+y^2+z^2}},\frac{z}{\sqrt{\mathrm{x}^2+y^2+z^2}}\right)\times\mathbf{v}=\frac{\mathbf{r}\times\mathbf{v}}{|\mathbf{r}|}$$

SVAR: $\dfrac{\mathbf{r}\times\mathbf{v}}{|\mathbf{r}|}$

11.11b. Beräkna

$$\operatorname{rot}\left(|\mathbf{r}|\,\mathbf{r}\right)$$

Lösning:

Resultatet från övning 11.10b ger att

$$\operatorname{rot}\left(|\mathbf{r}|\,\mathbf{r}\right)=\left(\operatorname{grad}|\mathbf{r}|\right)\times\mathbf{r}+|\mathbf{r}|\underbrace{\operatorname{rot}\mathbf{r}}_{=0}=\left(\operatorname{grad}\sqrt{\mathrm{x}^2+y^2+z^2}\right)\times\mathbf{r}=$$

$$=\left(\frac{x}{\sqrt{\mathrm{x}^2+y^2+z^2}},\frac{y}{\sqrt{\mathrm{x}^2+y^2+z^2}},\frac{z}{\sqrt{\mathrm{x}^2+y^2+z^2}}\right)\times\mathbf{r}=\frac{\mathbf{r}\times\mathbf{r}}{|\mathbf{r}|}=\mathbf{0}$$

eftersom

$$\operatorname{rot}\mathbf{r}=\begin{vmatrix}\hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x & y & z\end{vmatrix}=\mathbf{0}$$

SVAR: 0

11.11c. Beräkna

$$\operatorname{rot}\left(f(|\mathbf{r}|)\mathbf{r}\right)$$

där $f(\mathbf{r})$ är en $\mathbf{R}\to\mathbf{R}$ funktion med kontinuerliga derivator.

Lösning:

Resultatet från övning 11.10b ger att

$$\operatorname{rot}\left(f(|\mathbf{r}|)\mathbf{r}\right)=\left(\operatorname{grad}f(|\mathbf{r}|)\right)\times\mathbf{r}+f(\mathbf{r})\underbrace{\operatorname{rot}\mathbf{r}}_{=0}=$$

$$=\left\lceil\text{Kedjeregeln}\right\rceil=\left(f'(|\mathbf{r}|)\operatorname{grad}\,|\mathbf{r}|\right)\times\mathbf{r}=f'(|\mathbf{r}|)\frac{\mathbf{r}\times\mathbf{r}}{|\mathbf{r}|}=\mathbf{0}$$

SVAR: 0
