AN APPLICATION of LINEAR ALGEBRA

When retrieved from the Internet, digital images take a considerable amountof time to download and use a large amount of computer memory.

The basic idea behind this method of compression is to treat a digital image as an array of numbers i.e., a matrix. Each image consists of a fairly large number of little squares called pixels (picture elements). The matrix corresponding to a digital image assigns a whole number to each pixel. For example, in the case of a 256x256 pixel gray scale image, the image is stored as a 256x256 matrix, with each element of the matrix being a whole number ranging from 0 (for black) to 225 (for white). The JPEG compression technique divides an image into 8x8 blocks and assigns a matrix to each block. One can use some linear algebra techniques to maximize compression of the image and maintain a suitable level of detail.

Images are comprised of pixels represented by numbers

$$\begin{bmatrix} 64 & 2 & 3 & 61 & 60 & 6 & 7 & 57 \\ 9 & 55 & 54 & 12 & 13 & 51 & 50 & 16 \\ 17 & 47 & 46 & 20 & 21 & 43 & 42 & 24 \\ 40 & 26 & 27 & 37 & 36 & 30 & 31 & 33 \\ 32 & 34 & 35 & 29 & 28 & 38 & 39 & 25 \\ 41 & 23 & 22 & 44 & 45 & 19 & 18 & 48 \\ 49 & 15 & 14 & 52 & 53 & 11 & 10 & 56 \\ 8 & 58 & 59 & 5 & 4 & 62 & 63 & 1 \end{bmatrix}$$

What is used in "COMPRESSING" and "DECOMPRESSING":

Suppose

$$r = \begin{bmatrix} 420 & 680 & 448 & 708 & 1260 & 1420 & 1600 & 1600 \end{bmatrix}$$

is one row of an 8x8 image matrix

By performing the following operations on the entries of the vector r:
1. Divide the entries of r into four pairs: (420, 680), (448, 708), (1260, 1410), (1600, 600).
2. Form the average of each of these pairs

One forms the first four entries of the next step vector r1.

$$r_1 = \begin{bmatrix} 550 & 578 & 1340 & 1600 & -130 & -130 & -80 & 0 \end{bmatrix}$$

Note that the vector r1 can be obtained from r by multiplying r on the right by the matrix:

$$W_1 = \begin{bmatrix} 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & -1/2 \end{bmatrix}$$

The first four coefficients of r1 are called the approximation coefficients and the last four entries are called the detail coefficients.

Iterating this process one gets:

$$r_2 = \begin{bmatrix} 564 & 1470 & -14 & -130 & -130 & -130 & -80 & 0 \end{bmatrix}$$

Here the vector r2 can be obtained from r1 by multiplying r1 on the right by the matrix:

$$W_2 = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and r3 can be obtained from r1 by multiplying r2 on the wright by the matrix:

$$W_3 = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

r3 can be obtained from r1 by multiplying r2 on the wright by the matrix:

$$r_3 = W_1 W_2 W_3 r$$

Let:

$$W = W_1 W_2 W_3 = \begin{bmatrix} 1/8 & 1/8 & 1/4 & 0 & 1/2 & 0 & 0 & 0 \\ 1/8 & 1/8 & 1/4 & 0 & -1/2 & 0 & 0 & 0 \\ 1/8 & 1/8 & -1/4 & 0 & 0 & 1/2 & 0 & 0 \\ 1/8 & 1/8 & -1/4 & 0 & 0 & -1/2 & 0 & 0 \\ 1/8 & -1/8 & 0 & 1/4 & 0 & 0 & 1/2 & 0 \\ 1/8 & -1/8 & 0 & 1/4 & 0 & 0 & -1/2 & 0 \\ 1/8 & -1/8 & 0 & -1/4 & 0 & 0 & 0 & 1/2 \\ 1/8 & -1/8 & 0 & -1/4 & 0 & 0 & 0 & -1/2 \end{bmatrix}$$

Note the following:
The columns of the matrix W1 form an orthogonal subset of R8 (the vector space of dimension 8 over R); that is these columns are pair wise orthogonal (try their dot products). Therefore, they form a basis of R8. As a consequence, W1 is invertible. The same is true for W2 and W3.
As a product of invertible matrices, W is also invertible and its columns form an orthogonal basis of R8.
The inverse of W is given by:

$$W^{-1} = W_3^{-1} W_2^{-1} W_1^{-1}$$

Suppose that A is the matrix corresponding to a certain image. The Haar transform is carried out by performing the above operations on each row of the matrix A and then by repeating the same operations on the columns of the resulting matrix. The row-transformed matrix is AW. Transforming the columns of AW is obtained by multiplying AW on the left by the matrix W^T (the transpose of W).

Thus, the Haar transform takes the matrix A and stores it as WTAW. Let S denote the transformed matrix:

$$S = W^T A W.$$

Using the properties of inverse matrix, we can retrieve our original matrix:

$$A = (W^T)^{-1} S W^{-1} = (W^{-1})^T S W^{-1}.$$

EXAMPLE

This is a picture of a famous mathematician: Emmy Noether compressed in different ways