Problem 73# 3.7 Solution

Problem 7: Prove

(*) $\qquad 2^{-x} + 2^{-1/x} \le 1 \quad \text{for} \quad x > 0.$

Set $F(x) = 2^{-x} + 2^{-1/x}$. Since $F(x) = F(1/x)$ it is sufficient to prove (*) for $0 < x \le 1$.

Rewrite (*) as follows:

(*) $\quad \Leftrightarrow \quad 2^{-1/x} \le 1 - 2^{-x} \quad \Leftrightarrow \quad \ln(2^{-1/x}) \le \ln(1 - 2^{-x}) \quad \Leftrightarrow$

$\Leftrightarrow \quad f(x) \le 0$, where $f(x) = -\dfrac{\ln 2}{x} - \ln(1 - 2^{-x})$.

Now, $f(1) = 0$ and we are done if we can prove that $f(x)$ increases in $0 < x < 1$.

$$f'(x) = \frac{\ln 2}{x^2} - \frac{\ln 2 \cdot 2^{-x}}{1 - 2^{-x}} = \frac{\ln 2}{x^2} - \frac{\ln 2}{2^x - 1}.$$

Hence, $f'(x) \ge 0 \quad \Leftrightarrow \quad x^2 \le 2^x - 1 \quad \Leftrightarrow \quad g(x) = x^2 + 1 - 2^x \le 0.$

But $g''(x) = 2 - \ln^2 2 \cdot 2^x > 0$ in $0 < x < 1$ and $g(0) = g(1) = 0$.

Then $g(x) \le 0$ by convexity in $0 < x < 1$, so $f'(x) \ge 0$ there and hence $f(x)$ increases. QED.

GJ