**Problem73# 3.10 Solution B**

Lemma: $f(x) = \dfrac{\log(1+x)}{x}$ is convex on $(0,\infty)$. (Assume this).

Then $\dfrac{\log(1+\sum p_j a_j)}{\sum p_j a_j} \quad \le \quad \sum p_j \dfrac{\log(1+a_j)}{a_j}$,

where $p_j > 0, \quad \sum p_j = 1$.

Now, let $p_j = \dfrac{a_j}{A}, \quad A = \sum_{j=1}^{n} a_j, \quad B = \sum_{j=1}^{n} a_j^2, \quad$ i.e. $\sum p_j a_j = B/A$.

We get $\dfrac{\log(1+\frac{B}{A})}{B/A} \le \left(\sum \dfrac{a_j \log(1+a_j)}{a_j}\right) \dfrac{1}{A}$,

whence $\sum \log(1+a_j) \ge \dfrac{A^2}{B} \log(1+\dfrac{B}{A}), \qquad$ QED.

Compact calculus proof of lemma:

$$f = \frac{\log(1+x)}{x}, \qquad f' = -\frac{1}{x^2}\log(1+x) + \frac{1}{x(1+x)}$$
$$f'' = \frac{2\log(1+x)}{x^3} - \frac{3x+2}{x^2(1+x)^2} = \frac{1}{x^3}\left(2\log(1+x) - \frac{3x^2+2x}{(1+x)^2}\right) =$$
$\dfrac{1}{x^3}g(x), \qquad g(0) = 0,$ enough to show: $g' > 0$ for $x > 0$.

$g' = \dfrac{2}{1+x} - \dfrac{2+4x}{(1+x)^3} = \dfrac{2x^2}{(1+x)^3} > 0.$ Hence $f'' > 0$ on $x > 0$. QED.

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