Problem 73# 2.3 Solution

This proof has been attributed to Diaz-Metcalf.

Rewriting the given inequalities:
$a \leq a_i \leq A, \qquad b \leq b_i \leq B$ to

$ab_i \leq a_i B, \qquad a_i b \leq A b_i,$
$(a_i B - ab_i)(Ab_i - a_i b) \geq 0,$
$aAb_i^2 + bBa_i^2 \leq a_i b_i (AB + ab)$ for all i.

Summing:
$aA\sum b_i^2 + bB\sum a_i^2 \leq (ab + AB)(\sum a_i b_i)$

But since (geometric/arithmetic inequality):

$aA\sum b_i^2 + bB\sum a_i^2 \geq 2\sqrt{aA\sum b_i^2 \cdot bB\sum a_i^2}$ we get

$$\sqrt{\sum a_i^2} \cdot \sqrt{\sum b_i^2} \leq \frac{ab + AB}{2\sqrt{abAB}} \sum a_i b_i$$

and hence the following which is equivalent to the given inequality:

$$\sqrt{\sum a_i^2} \cdot \sqrt{\sum b_i^2} \leq \frac{1}{2}\left(\sqrt{\frac{ab}{AB}} + \sqrt{\frac{AB}{ab}}\right) \sum a_i b_i.$$