Problem73# 1.6 Solution B

For simplicity we begin to consider the case of two kind of stamps, 1 p. and 2 p. Then a_n, the number of n p. postage sequences, can be computed recursively by dividing these sequences into two groups, S_1 that starts with a 1p. stamp and S_2 that starts with a 2p. stamp. Now the number of S_1 sequences must be equal to the number of $n-1$ postage sequences that can follow the starting 1p. stamp, i.e. a_{n-1}. Similarly the number of S_2 sequences must be a_{n-2}. (Assume here for convenience that $a_0 = 1$. Also a_1 must be 1). Hence $a_n = a_{n-1} + a_{n-2}, \quad n \geq 2, \quad a_0 = a_1 = 1$.

The $a_n : s$ are of course recognized as the Fibonacci sequence which begins $1, 1, 2, 3, 5, 8, 13, ...$

The function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ can be recovered as follows:

Note that $xf(x) + x^2 f(x) = \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+2} =$

$a_0 x + \sum_{n=2}^{\infty} (a_{n-1} + a_{n-2}) x^n = a_0 x + \sum_{n=2}^{\infty} a_n x^n =$

$a_0 x + (f(x) - a_0 - a_1 x) = x + f(x) - 1 - x = f(x) - 1.$

And so, $f(x)(1 - x - x^2) = 1, \qquad f(x) = \dfrac{1}{1 - x - x^2}.$

(Note that a power series expansion of this $f(x)$ will yield $f(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + ...$, i.e. the expected Fibonacci coefficients.)

A repetition of the above arguments will take care of the given problem. With 1p. , 2p. , 3p. and 4p. stamps we get the recursion formula $a_n = a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}$. Accordingly the corresponding function satisfies $xf(x) + x^2 f(x) + x^3 f(x) + x^4 f(x) = f(x) - 1$ which gives the solution

$f(x) = \dfrac{1}{1 - x - x^2 - x^3 - x^4}.$