Problems # 2 1973. INEQUALITIES

1. Give an algebraic proof of the Cauchy inequality by showing that

$$(\sum_1^n x_i^2)(\sum_1^n y_i^2) - (\sum_1^n x_i y_i)^2$$

can be expressed as a sum of squares of real quantities.

2. Let $E \subset R^n$ denote the "ellipsoid" :
$E = \{(x_1, ..., x_n) : \quad \sum_{i=1}^n \frac{x_i^2}{a_i^2} \leq 1\}$
whose semiaxes are given positive numbers $a_1, ..., a_n$.
Prove (analytically) that if $(\xi_1, ..., \xi_n)$ is a point on the surface of E (that is, $\sum_{i=1}^n (\xi_i^2/a_i^2) = 1$) the "hyperplane" $\sum_{i=1}^n \frac{\xi_i x_i}{a_i^2} = 1$ meets E <u>only</u> in the point $(\xi_1, ..., \xi_n)$.

3. Let a, A, b, B be positive, $\quad a < A, \quad b < B$.
Suppose $a \leq a_i \leq A, \quad b \leq b_i \leq B \quad (i = 1, 2, ..., n)$. <u>Prove:</u>

$$\frac{(a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2)}{(a_1 b_1 + a_2 b_2 + ... + a_n b_n)^2} \quad \leq \quad \left(\frac{\sqrt{\frac{AB}{ab}} + \sqrt{\frac{ab}{AB}}}{2} \right)$$

4. Give a "convexity" proof of Hölder's inequality

$$\sum_i^n a_i b_i \quad \leq \quad (\sum_i^n a_i^{1/\theta})^\theta (\sum_i^n b_i^{1/(1-\theta)})^{1-\theta}$$

(here $a_i, b_i \geq 0; \quad 0 < \theta < 1$) by considering the centroid of n masses placed on the curve $y = x^\theta$.

5. Prove the "cyclic inequality"

$$\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+...+\frac{x_{n-1}}{x_n+x_1}+\frac{x_n}{x_1+x_2}\geq\frac{n}{2}$$

for $n = 3, 4, 5, 6$.
(The inequality is true for $n \leq 10$, and false for $n = 14$!)

6. <u>Prove</u> $\sqrt[n]{(a_1+b_1)(a_2+b_2)...(a_n+b_n)} \quad \geq \quad \sqrt[n]{a_1a_2...a_n}+\sqrt[n]{b_1b_2...b_n}$
$(a_i, b_i \geq 0$).

7. <u>Prove:</u> for $x, y, z > 0$ and λ real,
$x^\lambda(x-y)(x-z)+y^\lambda(y-z)(y-x)+z^\lambda(z-x)(z-y) \quad \geq \quad 0.$