Matematiska Institutionen
KTH

Solutions to homework number 5 to SF2736, fall 2012.

Please, deliver this homework at latest on Monday, December 3.

The homework must be delivered individually, and, in general, just hand-written notes are accepted. You are free to discuss the problems below with your class mates, but you are not allowed to copy the solution of another student.

1. (0.2p) Use the technique with generating functions to find explicit expressions for the numbers a_n, $n = 0, 1, 2, ...$, if this sequence of numbers satisfies

$$a_n = 3a_{n-2} + 2a_{n-3}, \qquad n = 3, 4, 5...$$

and $a_0 = 2$, $a_1 = 0$ and $a_2 = 7$.

Solution. Define $A(t) = \sum_{n=0}^{\infty} a_n t^n$. As

$$a_n t^n = 3t^2 a_{n-2} t^{n-2} + 2t^3 a_{n-2} t^{n-3}$$

for $n = 3, 4, \ldots$, we get by summing up things that

$$A(t) - a_2 t^2 - a_1 t - a_0 = 3t^2 (A(t) - a_0) + 2t^3 A(t)$$

and thus, using the facts that $a_0 = 2$, $a_1 = 0$ and $a_2 = 7$, we get

$$A(t) = \frac{2 + t^2}{1 - 3t^2 - 2t^3}$$

Expanding, first in partial fractions

$$A(t) = \frac{2 + t^2}{(1 - 2t)(1 + t)^2} = \frac{1}{1 - 2t} + \frac{1}{(1 + t)^2}$$

and then in power series

$$A(t) = \frac{1}{1 - 2t} + \frac{1}{(1 + t)^2} = \sum_{n=0}^{\infty} 2^n t^n + \sum_{n=0}^{\infty} (-1)^n (n + 1) t^n =$$

$$= \sum_{n=0}^{\infty} (2^n + (-1)^n (n + 1)) t^n$$

So

Answer: $a_n = 2^n + (-1)^n (n + 1)$.

2. (0.2p) How many distinct necklaces with 12 beads can you form by using just red, green and blue beads.

   Solution. We use the lemma of Burnside and thus consider the automorphism group of the necklace. Either we can rotation the necklace, or turn it over, mirror it in a diagonal. The automorphism group consists of 24 elements. We have to consider the following table, in which the beads in the necklace are enumerated in the order they appear by 1, 2, 3, ..., 12:

$\varphi \in \mathrm{Aut}(G)$	$\lvert\mathrm{Fix}(\varphi)\rvert$
id.	3^{12}
$\varphi = (1\ 2\ 3\ \ldots\ 12)$	3
$\varphi^2 = (1\ 3\ 5\ 7\ 9\ 11)(2\ 4\ 6\ 8\ 10\ 12)$	3^2
$\varphi^3 = (1\ 4\ 7\ 10)(2\ 5\ 8\ 11)(3\ 6\ 9\ 12)$	3^3
$\varphi^4 = (1\ 5\ 9)(2\ 6\ 10)(3\ 7\ 11)(4\ 8\ 12)$	3^4
$\varphi^5 = (1\ 6\ 11\ \ldots\ 12)$	3
$\varphi^6 = (1\ 7)(2\ 8)\cdots(6\ 12)$	3^6
$\varphi^7 = (\varphi^5)^{-1}$	3
$\varphi^8 = (\varphi^4)^{-1}$	3^4
$\varphi^9 = (\varphi^3)^{-1}$	3^3
$\varphi^{10} = (\varphi^2)^{-1}$	3^2
$\varphi^{11} = (\varphi^1)^{-1}$	3
$\psi_1 = (1)(7)(2\ 12)(3\ 11)(4\ 10)(5\ 9)(6\ 8)$	3^7
ψ_2	3^7
ψ_3	3^7
ψ_4	3^7
ψ_5	3^7
ψ_6	3^7
$\gamma_1 = (1\ 12)(2\ 11)(3\ 10)(4\ 9)(5\ 8)(6\ 7)$	3^6
γ_2	3^6
γ_3	3^6
γ_4	3^6
γ_5	3^6
γ_6	3^6

   as we require that beads in the same cycle must have the same color, if they shall be fixed by a permutation of the beads.

Answer: $\frac{1}{24}(3^{12} + 6 \cdot 3^7 + 7 \cdot 3^6 + 2 \cdot 3^4 + 2 \cdot 3^3 + 2 \cdot 3^2 + 4 \cdot 3)$

3. We consider binary words $\bar{c} = a_1 a_2 \ldots a_{11}$ of length 11 and where $a_i \in Z_2$. Let $\mathbf{H}$ denote the matrix

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Let C be the set of words $\bar{c}$ such that $\mathbf{H}\bar{c}^T = \bar{0}^T$, if we count modulo 2.

(a) (0.2p) The code C is an e-error-correcting code. Find e.

Solution. Any combination of columns no 8, 9 and 10 gives columns with four ones, and if any of the columns so appearing is combined with column number 11, we get a column with three ones. So a linear combination of columns will not be equal to the zero column unless five columns are involved, in a non-trivial way, in the linear combination. This implies that the minimum weight of the code C is equal to 5. As $5 = 2e + 1$ if $e = 2$ the code is
Answer: 2-error-correcting.

(b) (0.2p) Can the word 01101001010 be corrected. If "yes" correct it. If "no" explain why.

Solution. We get that

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

which is the sum of column number 9 and 11. Hence we correct the word in positions 9 and 11 and get

Answer: 01101001111

(c) (0.2p) Find the number of words in Z_2^{11} that cannot be corrected by this code C.

Solution. The code C is the null space of the matrix $\mathbf{H}$, which have rank 7. Hence the number of words in C is equal to

$$|C| = 2^{11-7} = 16.$$

The number of words in a sphere of radius 2 around a code word $\bar{c}$ is equal to

$$|\mathrm{S}_2(\bar{c})| = 1 + \binom{11}{1} + \binom{11}{2} = 1 + 11 + 55 = 67.$$

The number of words of length 11 that belong to 2-spheres around words is thus equal to

$$16 \cdot 67 = 1072.$$

The number of words that cannot be corrected is thus

Answer: $2^{11} - 1072 = 976$