Matematiska Institutionen
KTH

Solutions to homework number 3 to SF2736, fall 2012.

Please, deliver this homework at latest on Tuesday, November 20.

The homework must be delivered individually, and, in general, just hand-written notes are accepted. You are free to discuss the problems below with your class mates, but you are not allowed to copy the solution of another student.

1. (0.2p) In how many ways can the elements in set $\{1, 2, 3, \ldots, 15\}$ be distributed into the three sets A, B and C such that

$$|A \cap B| = |A \cap C| = |B \cap C| = 1.$$

Solution. There are two cases to treat:

Case 1. $A \cap B \cap C = \emptyset$. We first choose elements to the subsets $A \cap B$, $A \cap C$ and $B \cap C$. This can be done in $n_1 = 15 \cdot 14 \cdot 13$ distinct ways. For each of the remaining 12 elements there are three possibilities, either to be placed in the set $A \setminus (B \cup C)$, or in the set $B \setminus (A \cup C)$, or $C \setminus (B \cup A)$, so in total $n_2 = 3^{12}$ possibilities.

Case 2. $A \cap B \cap C \neq \emptyset$. Choose one element to the set $A \cap B \cap C$, there are 15 elements to choose among so in total $n'_1 = 15$ possibilities. As in the previous case we get that the other 14 elements can be distributed in $n'_2 = 3^{14}$ ways.

Answer: $15 \cdot 14 \cdot 13 \cdot 3^{12} + 15 \cdot 3^{14}$.

2. (0.2p) In how many ways can the set $\{1, 2, 3, \ldots, 10\}$ be divided into five non-empty subsets such that the elements 1, 2 and 3 belong to distinct subsets.

Solution. We label the sets with the elements 1, 2, 3 with the labels 1, 2 and 3, respectively, and the remaining two sets with the labels a and b. Let A denote the distribution of the remaining elements $\{4, 5, \ldots, 10\}$

such that the set a will be empty, and similarly for the set B. We use inclusion exclusion and get, as

$$|A \cup B| = |A| + |B| - |A \cap B|,$$

and, as we must dislabel the sets a and b, the

Answer: $\frac{1}{2}(5^7 - (2 \cdot 4^7 - 3^7))$.

3. (a) (0.1p) For how many 4-tuples (n_1, n_2, n_3, n_4) are the multinomial coefficients

$$\binom{12}{n_1, n_2, n_3, n_4}$$

defined, as the number of ways to partition a set of 12 elements into four labeled subsets, each with n_1, n_2, n_3 and n_4 elements, respectively.

Solution. An equivalent problem is to find the number of integer solutions to the equation

$$n_1 + n_2 + n_3 + n_4 = 12,$$

where $n_i \geq 1$ for $i = 1, 2, 3, 4$. An equivalent problem is to distribute 12 identical objects into four distinct and labeled boxes such that no box will be empty. It is known that the number of ways this can be done is

Answer:

$$\binom{12 - 4 + 3}{3} = \binom{11}{3} = 165.$$

If we accept that $n_i = 0$ then the answer will be 455.

(b) (0.2p) Find

$$\sum 3^{n_1}(-6)^{n_2} 9^{n_3}(-4)^{n_4} \binom{12}{n_1, n_2, n_3, n_4}$$

where the sum is taken over all 4-tuples (n_1, n_2, n_3, n_4) for which the multinomial coefficients above are defined. The answer must be given as an integer.

Solution. We use the formula

$$(x+y+z+u)^n = \sum \binom{n}{n_1, n_2, n_3, n_4} x^{n_1} y^{n_2} z^{n_3} u^{n_4}$$

where some of the integers n_i might be equal to zero. We obtain, with $x = 3$, $y = -6$, $z = 9$ and $u = -4$, the answer

Answer:

$$(3-6+9-4)^{12} = 4096.$$

4. (0.3p) A class consists of eight girls and nine boys. In how many ways can the children be distributed in the three class rooms D2, E2 and Q2 in such a way that each of the class rooms contains at least two boys and two girls.

Solution. We divide into cases, and consider the distribution of girls and boys separately. Girls first.

Case 1. One classroom contains 4 girls and the remaining 2 girls each. So first choose which classroom will contain 4 girls and then distribute the girls. The number of ways this can be done is

$$3\binom{8}{4,2,2}.$$

Case 2. Two classroom contain 3 girls each and one classroom 2 girls. The number of ways the girls can be distributed in this case is

$$3\binom{8}{3,3,2}.$$

Similarly for the boys, but three cases. We get

Answer:

$$(3\binom{8}{4,2,2} + 3\binom{8}{3,3,2}) \cdot (3\binom{9}{5,2,2} + \binom{3}{1,1,1}\binom{9}{4,3,2} + \binom{9}{3,3,3}).$$