Matematiska Institutionen
KTH

Solutions to homework number 2 to SF2736, fall 2012.

Please, deliver this homework at latest on Wednesday, November 14.

1. (0.1p) Let $\mathcal{R}$ be an equivalence relation on a set A. Assume that $|A| = 45$ and assume that $\mathcal{R}$ induces a partition of A into five equivalence classes of equal size. Find $|\mathcal{R}|$.

   Solution. Each equivalence class C_i, $i = 1, 2, 3, 4, 5$, consist of $45/5 = 9$ elements
   $$C_i = \{a_{i1}, a_{i2}, \ldots, a_{i9}\}.$$
   As $\mathcal{R}$ is an equivalence relation we have that for each $i = 1, 2, 3, 4, 5$, and each pair $j, j' \in \{1, 2, \ldots, 9\}$, it is true that
   $$a_{ij} \sim a_{ij'}$$
   or expressed in another way as
   $$(a_{i,j}, a_{ij'}) \in \mathcal{R}.$$
   So each equivalence class contributes with $9 \cdot 9 = 81$ elements to the set $\mathcal{R}$. As $i \neq i'$ implies that $a_{ij} \not\sim a_{i'j'}$, these five contributions to the set $\mathcal{R}$ includes all contributions. Thus

   Answer: $5 \cdot 81 = 405$.

2. (0.2p) Let $\mathcal{R}$ be a relation on a set A which is both transitive and symmetric. Define
   $$C_a = \{x \in A \mid a\mathcal{R}x\}.$$
   Is the following true
   $$C_a \neq C_b \qquad \Longrightarrow \qquad C_a \cap C_b = \emptyset.$$

   Solution. Yes. We first note that a set $C_a \neq \emptyset$ if and only if $a\mathcal{R}x$ for at least one element $x = x_a \in A$. We thus consider the subset A'

of A consisting of those elements in A that are related to at least one element $x \in A$. The restriction of $\mathcal{R}$ to A' is an equivalence relation on A', as symmetry and transitivity gives that

$$a\mathcal{R}x_a \quad \Rightarrow \quad \begin{cases} a\mathcal{R}x_a \\ x_a\mathcal{R}a \end{cases} \quad \Rightarrow \quad a\mathcal{R}a.$$

The set C_a, as well as any set C_b, will thus be an equivalence class in A'. We know that distinct equivalence classes, (to equivalence relations), are mutually disjoint.

An alternative solution:

Assume that y is any element in C_a and that $x \in C_a \cap C_b$. Then by symmetry and transitivity we have that

$$\begin{cases} a & \sim & y \\ a & \sim & x \\ b & \sim & x \end{cases} \quad \Rightarrow \quad \begin{cases} y & \sim & a \\ a & \sim & x \\ x & \sim & b \end{cases} \quad \Rightarrow \quad y \sim b \quad \Rightarrow \quad b \sim y,$$

that is, $y \in C_b$. We have thus proved

$$C_a \cap C_b \neq \emptyset \qquad \Longrightarrow \qquad C_a \subseteq C_a.$$

Similarly we can prove that

$$C_a \cap C_b \neq \emptyset \qquad \Longrightarrow \qquad C_b \subseteq C_a.$$

These two implications thus imply that

$$C_a \cap C_b \neq \emptyset \qquad \Longrightarrow \qquad C_a = C_a.$$

3. Let $A = \{1, 2, \ldots, 9\}$ and let $\mathcal{R}$ be the following relation on A:

$$\mathcal{R} = \{(1,1), (2,3), (3,5), (7,6), (7,7), (8,9)\}.$$

   (a) (0.1p) Find the smallest equivalence relation that contains $\mathcal{R}$.

   Solution. The equivalence relation that has the following five equivalence classes:

$$C_1 = \{1\},\ C_2 = \{2,3,5\},\ C_4 = \{4\},\ C_6 = \{6,7\},\ C_8 = \{8,9\}.$$

(b) (0.2p) Find the number of equivalence relations that contain $\mathcal{R}$.

Solution. The equivalence classes to an equivalence relation that contains $\mathcal{R}$ must be unions of a choice of the five equivalence classes above. So an equivalent problem is to find the number of ways to partition the set $\{C_1, C_2, C_4, C_6, C_8\}$ into subsets. As we now have learned about Stirling numbers, let us use that description.

The answer will be given by

$$\mathrm{S}(5,5) + \mathrm{S}(5,4) + \mathrm{S}(5,3) + \mathrm{S}(5,2) + \mathrm{S}(5,1).$$

Calculating these Stirling numbers using the recursion

$$\mathrm{S}(n,k) = \mathrm{S}(n-1,k-1) + k\mathrm{S}(n-1,k),$$

we get the sum

$$1 + 10 + 25 + 15 + 1 = 52.$$

Answer: 52

4. (0.1p) Find and describe a bijective map that maps the set of real numbers in the open interval $(3,7)$ onto the interval $(2,3)$.

Solution. Linear maps $x \mapsto ax + b$ are bijective maps. If we choose a and b such that

$$\begin{cases} 3a + b = 2 \\ 7a + b = 3 \end{cases}$$

then the interval $(3,7)$ is mapped bijectively onto the interval $(2,3)$.

The system above is linear and thus easily solved, E.G. by using Gauss elimination.

Answer: The map $x \mapsto 0.25x + 1.25$.

5. (0.3p) Assume that A is a given countable infinite set and let B be the set of all real numbers x that are solutions to some polynomial equation

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0,$$

where $a_i \in A$, for $i = 0, 1, \ldots, n$. Is the set B countable infinite?

Solution. In case $0 \in A$ and for those that regard $0 = 0$ as a polynomial equation in the indeterminate x, then, as every real number satisfies this equation, their answer in this case will be "No".

But here we do not regard "$0 = 0$" as a polynomial equation.

We first show that the set of all solutions must be infinite. Let a_1 be a fixed non-zero element in A. Let S denote the set of solutions to the equations

$$ax = a_1, \qquad \text{for} \qquad a \in A.$$

Then

$$S = \{\frac{a_1}{a} \mid a \in A \setminus \{0\}\},$$

and consequently S is countable infinite.

By a proof, similar to the proof of the fact that the set of rational numbers is countable infinite, one may prove that the direct product $A^2 = A \times A$ is countable infinite, and then inductively that $A^n = A \times A \times \cdots \times A$ is countable. We now note that the $n : th$ degree equations, with coefficients in A, are in one to one correspondence with the elements of the direct product A^{n+1}.

Again, by a proof, similar to the proof of the fact that the set of rational numbers is countable infinite, one may prove that the union of an infinite set of mutually disjoint countable infinite sets is countable infinite. This implies that the set of all polynomial equations with coefficients in A is an infinite countable set:

$$\{EQ_1, EQ_2, EQ_3, \ldots\}.$$

The number of solutions of each such equation is finite. So we can now enumerate all solutions to all these equations by first the enumeration of the solutions of EQ_1, then continue our enumeration with the solutions of EQ_2, etc. In our enumeration of the solutions, we do not give a number to those real numbers already enumerated by us, we just skip them.