Matematiska Institutionen
KTH

Solutions to some of the problems in a the typical exam to the course Discrete Mathematics, SF2736, fall 12.

Observe:

1. Nothing else than pencils, rubbers, rulers and papers may be used.
2. Bonus points from the homeworks will be added to the sum of points on part I.
3. Grade limits: 13-14 points will give Fx; 15-17 points will give E; 18-21 points will give D; 22-27 points will give C; 28-31 points will give B; 32-36 points will give A.

Part I

1.
2.
3.
4.
5.

Part II

6. (a) (1p) Prove that $\gcd(a, b)$ divides $\operatorname{lcm}(a, b)$ for all integers a and b.

   Solution.

   $$\gcd(a,b) \mid a \quad \text{and} \quad a \mid \operatorname{lcm}(a,b) \qquad \Longrightarrow \qquad \gcd(a,b) \mid \operatorname{lcm}(a,b)$$

   (b) (2p) Find the number of sets $\{x, y\}$ such that x and y are non negative integers and

   $$\gcd(x,y) = 60 \qquad \text{and} \qquad \operatorname{lcm}(x,y) = 12600.$$

   Solution. We get

   $$60 = 2^2 \cdot 3 \cdot 5 \qquad \text{and} \qquad 12600 = 2^3 \cdot 3^2 \cdot 5^2 \cdot 7.$$

As then

$$60 \mid x \qquad \text{and} \qquad x \mid 12600 \qquad \Longrightarrow \qquad x = 2^{e_1} \cdot 3^{e_2} \cdot 5^{e_3} \cdot 7^{e_4},$$

where

$$(e_1, e_2, e_3, e_4) \in \{\{2,3\} \times \{1,2\} \times \{1,2\} \times \{0,1\}\}.$$

and similarly for $y = 2^{f_1} \cdot 3^{f_2} \cdot 5^{f_3} \cdot 7^{f_4}$. To get that the least common multiple of x and y is 12600 and the greatest common divisor equal to 60 it is necessary and sufficient that

$$\{e_1, f_1\} = \{2,3\}, \quad \{e_2, f_2\} = \{1,2\},$$

$$\{e_3, f_3\} = \{1,2\}, \quad \{e_4, f_4\} = \{0,1\},$$

This gives in total $2^4 = 16$ possibilities to get pairs (x, y) satisfying the required conditions. However, as we asked for the number of sets $\{x, y\}$, and $x \neq y$ in every feasibly pair, we have

Answer: 8.

7.

8.

Part III

9.

10. (5p) Let $\varphi(n)$ denote the number of integers $1 \leq m \leq n$ that are relatively prime to n. Show that for all positive integers n, and all prime numbers p, it is true that n divides $\varphi(p^n - 1)$.

Solution. Let $\mathrm{U}(Z_m)$ denote the group of all invertible elements in the ring Z_m. Then

$$\gcd(p, p^n - 1) = 1 \quad \Longrightarrow \quad p \in \mathrm{U}(Z_{p^n-1}) \quad \Longrightarrow \quad o(p) \mid \varphi(p^n - 1),$$

as the size of the group $\mathrm{U}(Z_{p^n-1})$ is $\varphi(p^n - 1)$. But, $p^n = 1$ in the ring Z_{p^n-1} and $p^k < p^n - 1$ for $1 \leq k < n$ so $o(p) = n$ in the group $\mathrm{U}(Z_{p^n-1})$. Thus n divides $\varphi(p^n - 1)$.