Matematiska Institutionen
KTH

Solutions to the exam to the course Discrete Mathematics, SF2736, June 8, 2012, 08.00–13.00.

Observe:

1. You are not allowed to use anything else than pencils, rubber, rulers and papers at this exam.
2. To get the maximum number of points on a problem it is not sufficient to just give an answer, you must also provide explanations.
3. Bonus points from the homeworks will be added to the sum of the points on part I.
4. Grade limits: 13-14 points will give an Fx; 15-17 points will give an E; 18-21 points will give a D; 22-27 points will give a C; 28-31 points will give a B; 32-36 points will give an A.

Part I

1. (3p) Find all graphs G with the property that both G and its complement graph $\bar{G}$ are bipartite.

   Solution: Let X and Y denote the two parts of vertices formed by the bipartition of the set of vertices of G. The complete graphs on X and Y, respectively, are subgraphs of $\bar{G}$. Hence, it must be true that $|X| \in \{1, 2\}$, and $|Y| \in \{1, 2\}$. With a paper and a pencil it is now easy to draw all possible bipartite graphs satisfying these two conditions and judge which of those that have a bipartite complement.

2. (3p) Find $3513^{561} (\text{mod } 562)$.

   Solution: As $562 = 2 \cdot 281$ and 281 is a prime number we have that

   $$\varphi(562) = 562(1 - \frac{1}{2})(1 - \frac{1}{281}) = 280.$$

   We find that

   $$3513 \equiv_{562} 6 \cdot 562 + 141.$$

As $141 = 3\cdot 47$ we get that 141 and 562 are coprime. We may hence use the theorem of Euler:

$$3513^{561} \equiv_{562} 141^{2\cdot 280+1} \equiv_{562} (141^{\varphi(562)})^2 \cdot 141 \equiv_{562} 1 \cdot 141.$$

ANSWER: 141.

3. (3p) A package shall contain seven items. You can choose among red balls in a box and nine distinct books from a book shelf. How many distinct such packages can you form if the package must contain at least one book and at least one red ball.

Solution: You shall pick b balls and $7-b$ distinct books, for $b = 1, 2, ..., 6$. There is just one way to pick the balls, but $\binom{9}{7-b}$ ways to pick books. Thus the number of distinct packages will be

$$\binom{9}{1} + \cdots + \binom{9}{6} = \sum_{b=0}^{9} \binom{9}{b} - (\binom{9}{0} + \binom{9}{7} + \binom{9}{8} + \binom{9}{9}) = 2^9 - (1 + \frac{9\cdot 8}{1\cdot 2} + 9 + 1)$$

ANSWER: 565.

4. (3p) Find the number of ways to form a necklace consisting of 7 beads in the colors red, green and yellow.

Solution: We use the lemma of Burnside. The automorphism group G of the necklace consist of 14 elements. The elements of G and the number of colorings fixed by each such element are given in the table below. The beads are enumerated 1 2, ..., 7, the order in which they appear in the necklace.

$g \in G$	$\lvert\mathrm{Fix}(g)\rvert$
id.	3^7
$(1)(2\ 7)(3\ 6)(4\ 5)$	3^4
$(2)(3\ 1)(4\ 7)(6\ 6)$	3^4
$\vdots$	$\vdots$
$(7)(1\ 6)(2\ 5)(3\ 4)$	3^4
$\psi = (1\ 2\ 3\ 4\ 5\ 6\ 7)$	3
ψ^2	3
ψ^3	3
$\vdots$	$\vdots$
ψ^6	3

ANSWER:

$$\frac{1}{14}(3^7 + 7\cdot 3^4 + 6\cdot 3) = 198.$$

5. (3p) Does there exist an abelian (commutative) group G with two distinct subgroups H and K with a pair of cosets that coincide, that is, there are two elements a and b of G such that $aH = bK$?

Solution: The answer is "no". To prove this we note that

$$aH = bK \qquad \Longrightarrow \qquad H = a^{-1}bK.$$

The only coset to the subgroup K that is a subgroup is the trivial coset K. As H is a subgroup we hence get from the relation above that $a^{-1}bK$ must be the trivial coset K to K, that is $a^{-1}bK = K$. Consequently $H = K$.

Part II

6. (3p) Show that every connected graph with more than two vertices, and containing the same number of vertices of valency (degree) one as there are vertices of valency three, must have at least one cycle.

   Solution: We use the fact that the sum of the valencies is twice the number of edges. As the graph is connected there are no vertices of valency 0. Let n_i denote the number of vertices of valency i, and let e denote the number of edges and v the number of vertices. Since $n_1 = n_3$, we get that

   $$2e = n_1 + 2n_2 + 3n_3 + 4n_4 + \cdots = 2n_1 + 2n_2 + 2n_3 + 4n_4 + \cdots \geq 2(n_1 + n_2 + \cdots) = 2v.$$

   So $e \geq v$. A connected graph without a cycle is a tree. The number of edges of a tree is always one less the number of vertices of the tree. The graph can thus not be a tree and as being connected it must contain a cycle.

7. (a) (1p) Prove that if n and m are integers such that $310n + 147m = 1$ then n and m must be coprime, that is, $\gcd(n, m) = 1$.

   Solution: One easy point. If d divides both n and m, then d divides any linear combination $an + bm$ where a and b are integers. Hence, $\gcd(n, m)$ must divide $310n + 147m$, that is $\gcd(n, m)$ must divide 1.

   (b) (3p) Are there three non-zero, and pairwise coprime, integers n, m and k such that

   $$310n + 217m + 147k = 1?$$

   Solution: The answer is "yes". As a motivation for this answer we find appropriate integers n, m and k.

   First we find, by using the Euclidian algorithm, integers a and b such that $a147 + b310 = 2$.

   $$\begin{aligned} 310 &= 2 \cdot 147 + 16 \\ 147 &= 9 \cdot 16 + 3 \\ 16 &= 5 \cdot 3 + 1 \end{aligned}$$

   Hence

   $$1 = 16 - 5 \cdot 3 = 16 - 5(147 - 9 \cdot 16) = 46 \cdot 16 - 5 \cdot 147 = 46(310 - 2 \cdot 147) - 5 \cdot 147$$

and thus

$$2 = 92 \cdot 310 - 194 \cdot 147.$$

Next we find c and d such that $c \cdot 217 + d \cdot 2 = 1$, which is easy, take $d = 109$ and $c = -1$.

Now we have a preliminary solution to the given equation, which we get from the equality below:

$$-217 + 109(92 \cdot 310 + (-194) \cdot 147) = 1,$$

that is, $n' = 109 \cdot 92$, $m' = -1$ and $k' = -194 \cdot 109$ gives

$$310n' + 217k' + 147m' = 1.$$

As trivially $-310 \cdot 147 + 147 \cdot 310 = 0$ we get that with $n = n' - 147$, $m = -1$ and $k = k' + 310$ we still have

$$310n + 217m + 147k = 1.$$

It remains to prove that n and k are coprime. From the equality above we get, as $m = -1$

$$310n + 147k = 218.$$

If d divides both n and k then d divides 218, an integer with the prime factorization

$$218 = 2 \cdot 109.$$

As none of the prime numbers 2 and 109 divides both n and k we may conclude that n and k are coprime. Finally and trivially, -1 is coprime with every integer.

8. (4p) Eight boys and seven girls shall form three queues, that are labeled as queue no. 1, queue no. 2 and queue no 3. How many such distinct queues can you form if it is required that every queue must contain at least one boy and one girl, and the boys are placed either in the front or the rear of each queue?

Solution: We first find out how many boys and girls, respectively, there are in each queue. We can then assume that the objects are distinguishable by just their gender b, boy, and g, girl. Put one indistinguishable girl in each queue, remains 4 indistinguishable girls, and they can be distributed in $\binom{4+2}{2} = 15$ distinct ways. Similarly for the boys we get $\binom{5+2}{2} = 21$ distinct distributions of the boys. Next we decide whether the boys shall be placed at the front or the rear. Two choices for each queue makes in total $2^3 = 8$ possible ways.

Now it remains to distribute the individuals. Do that in some fixed order and place girls and boys in position after position. In total this gives $7! \cdot 8!$ ways.

The principle of multiplication now gives

ANSWER: $15 \cdot 21 \cdot 8 \cdot 7! \cdot 8!$

Part III

9. Let $\mathcal{S}_n$ denote the group that consists of all permutations of the elements in the set $\{1, 2, \ldots, n\}$.

   (a) (2p) Show that $\mathcal{S}_4$ has exactly four subgroups of size 6.

   Solution: The elements in a group of size six has either order 1, 2, 3 or 6. As 2 and 3 are prime numbers, a group element of order 2 or 3 must be a cycle. An element of order six can either be a 6-cycle or a product of a 2-cycle and a 3-cycle that are disjoint.

   For a product of a 2-cycle and a 3-cycle there are three possibilities:

   Case 1: The 2-cycle and 3-cycle are disjoint and the product $(a\ b)(c\ d\ e)$ will have order 6. This case cannot occur in $\mathcal{S}_4$ as five distinct elements are involved.

   Case 2: The 2-cycle and the 3-cycle share an element. The product will then be $(a\ b)(b\ c\ d) = (a\ b\ c\ d)$ which is an element of order 4. This case cannot occur in a subgroup with six elements.

   Case 3: The 2-cycle and the 3-cycle share two elements. In this case we get $(a\, b)(a\, b\, c) = (b\, c)$

   The conclusion is that in $\mathcal{S}_4$ just the Case 3 will appear and just three of the elements $\{1, 2, 3, 4\}$, elements in a set $\{a, b, c\}$ will be involved. The set of permutations of the elements in this set is a group with six elements.

   As there are four subsets containing exactly three elements, we get exactly four subgroups with six elements.

   (b) (2p) Find the number of subgroups of $\mathcal{S}_5$ of size 6.

   Solution: From the solution above we adopt that every subset of size 3 to $\{1, 2, 3, 4, 5\}$ defines a group with six elements, that is all permutations on this set with three elements. There are $\binom{5}{3} = 10$ subsets with three elements.

   The group $\mathcal{S}_5$ does not contain any 6-cycle. However Case 1 above gives elements of order six, and each such element generate a subgroup of order 6. In a cyclic group of order 6:

   $$< \varphi >= \{\varphi, \varphi^2, \varphi^3, \varphi^4, \varphi^5, \varphi^6 = \text{id.}\}$$

   there are two elements of order six, the elements φ and φ^5.

   To every choice of a 2-subset $A = \{a, b\}$ and a disjoint 3-subset $C = \{c, d, e\}$ to $\{1, 2, 3, 4, 5\}$ (with $A \cap C = \emptyset$), there are two permutations of order six

   $$\psi = (a\ b)(c\ d\ e), \qquad \text{and} \qquad \psi^5 = (a\ b)(c\ e\ d).$$

   Hence, the number of cyclic subgroups of order six to $\mathcal{S}_5$ is equal to the number of 2-subsets to $\{1, 2, 3, 4, 5\}$, that is, the number of cyclic subgroups of order six is $\binom{5}{2} = 10$.

   ANSWER: 20.

10. Suppose that G is a finite abelian group and that G_1, G_2, ..., G_n are subgroups of G satisfying

$$G = \bigcup_{i=1}^{n} G_i \qquad \text{and} \qquad i \neq j \quad \Rightarrow \quad G_i \cap G_j = \{0\}, \tag{1}$$

where 0 denotes the identity in G. Let C be the kernel of the map φ from $S = G_1 \times G_2 \times \cdots \times G_n$ to G defined by

$$\varphi((g_1, g_2, \ldots, g_n)) = g_1 + g_2 + \cdots + g_n.$$

(a) (2p) Show that we can define a distance function between the elements in S and an error-correcting procedure, in such a way that C is an 1-error-correcting code.

Solution: We define the distance between the words as the number of positions in which the words differ. To show that C is 1-error-correcting it then suffices to show that the minimum distance between any two words of C is at least 3. Note that

$$\mathrm{d}(\bar{x}, \bar{y}) = \mathrm{d}(\bar{x} - \bar{y}, \bar{0}).$$

As the map φ is "linear" we get that

$$\varphi(\bar{x}) = 0 = \varphi(\bar{y}) \qquad \Longrightarrow \qquad \varphi(\bar{x} - \bar{y}) = 0.$$

It hence suffices to check, which we do below, that the minimum weight of C is 3.

A word of weight one, e.g. the word $(g_1, 0, \ldots, 0)$, where $g_1 \in G_1 \setminus \{0\}$ can never belong to C as

$$g_1 + 0 + \cdots + 0 = g_1 \neq 0.$$

Similarly a word of weight 2, e.g the word $(g_1, g_2, 0, \ldots, 0)$, where $g_1 \in G_1 \setminus \{0\}$ and $g_2 \in G_2 \setminus \{0\}$, can never belong to C as

$$g_1 + g_2 + 0 + \cdots + 0 = 0 \qquad \Longrightarrow \qquad g_2 = -g_1 \in G_1,$$

which is an impossibility as $G_1 \cap G_2 = \{0\}$.

(b) (2p) Does C have any further error-correcting properties?

Solution: The answer is "yes". Every possible word in S is within distance one from a code word. Namely, let $\bar{h} = (h_1, \ldots, h_n)$ be any word of S and assume that

$$\varphi(\bar{h}) = h_1 + \cdots + h_n = h_i' \in G_i.$$

Then

$$\varphi(h_1, h_2, \ldots, h_i - h_i', \ldots, h_n) = h_i' - h_i' = 0,$$

and hence

$$(h_1, h_2, \ldots, h_{i-1}, h_i - h_i', h_{i+1} \ldots, h_n) \in C.$$

The word $(h_1, \ldots, h_n)$ are within distance one from the code word above.

(c) (2p) Find an abelian group G of size 27 and a family of subgroups of G having the property in Equation (1) above.

Solution: Let $G = Z_3^3$. The following enumeration of subgroups solves the problem

$$
\begin{array}{rcl}
G_1 & = & <(1,0,0)> \; = \; \{(0,0,0),(1,0,0),(2,0,0)\} \\
G_2 & = & <(0,1,0)> \; = \; \{(0,0,0),(0,1,0),(0,2,0)\} \\
G_3 & = & <(1,1,0)> \; = \; \{(0,0,0),(1,1,0),(2,2,0)\} \\
G_4 & = & <(1,2,0)> \; = \; \{(0,0,0),(2,1,0),(1,2,0)\} \\
G_5 & = & <(0,0,1)> \; = \; \{(0,0,0),(0,0,1),(0,0,2)\} \\
G_6 & = & <(1,0,1)> \\
G_7 & = & <(2,0,1)> \\
G_8 & = & <(0,1,1)> \\
G_9 & = & <(1,1,1)> \\
G_{10} & = & <(2,1,1)> \\
G_{11} & = & <(0,2,1)> \\
G_{12} & = & <(1,2,1)> \\
G_{13} & = & <(2,2,1)>
\end{array}
$$