Matematiska Institutionen
KTH

Exam to the course Discrete Mathematics, SF2736, May 31, 2013, 08.00–13.00.

Examiner: Olof Heden.

Observe:

1. You are not allowed to use anything else than pencils, rubber, rulers and papers at this exam.
2. To get the maximum number of points on a problem it is not sufficient to just give an answer, you must also provide explanations.
3. Bonus points from the homeworks will be added to the sum of the points on part I.
4. Grade limits: 13-14 points will give an Fx; 15-17 points will give an E; 18-21 points will give a D; 22-27 points will give a C; 28-31 points will give a B; 32-36 points will give an A.

Part I

1. (a) (1.5p) Find $\gcd(789, 1011)$.
   (b) (1.5p) Find $34^{50} (\text{mod } 78)$.
2. (3p) A connected graph G has 13 vertices, no multiple edges or loops. Which are the possibilities for the number of edges of G.
3. (3p) Let G be a cyclic group with 48 elements generated by the element g. There is one and only one subgroup H to G of size 12. Show that H is cyclic and find all generators of H.
4. (3p) Find the generating function to the sequence a_0, a_1, ..., where $a_0 = 2$, $a_1 = 3$ and $a_n = a_{n-1} - 6a_{n-2}$, for $n = 2, 3, \ldots$.
5. (3p) Nine distinct guys shall ride in a row on bicycles. How many distinct rows can then be formed if there are four green bikes, three red bikes and two yellow bikes, else the bikes are indistinguishable.

Continues on the opposite side.

Part II

6. (3p) Let p be a prime number. Show that if $p > 3$ then 24 divides $p^2 - 1$.

7. (4p) Let $\mathcal{S}_8$ denote the set of all permutations of the set $\{1, 2, \ldots, 8\}$. Write the permutation $\varphi = (1\ \ 3\ \ 5)(2\ \ 4\ \ 6\ \ 7)$ as a product of nine distinct permutations in $\mathcal{S}_8$ of which no two are inverses to each other, that is,

$$\varphi = \psi_1 \psi_2 \cdots \psi_9, \qquad \text{and} \qquad \psi_i \psi_j \neq \text{Id}.$$

   for $i \neq j$, or show that this is impossible, (if this happens to be the case).

8. (4p) In how many ways can the set $\{1, 2, \ldots, 8\}$ be divided into three mutually disjoint subsets each containing at least two elements.

Part III

9. A set C of binary words of length n is a covering 1-code in Z_2^n if every binary word of length n is within distance at most one from at least one word of C.

   (a) (1p) Show that the words in the set

$$C = \{0000, 1100, 1010, 1001, 0110, 0101, 0011, 1111\}$$

   is a covering 1-code in Z_2^4.

   (b) (1p) Are there any covering 1-codes in Z_2^4 with fewer words? An answer must contain a motivation.

   (c) (3p) A code C is linear if the difference between any two words of C also belongs to C. Find a construction of linear covering 1-codes, and discuss whether your construction is "the best possible".

10. (5p) (From Wilson: Introduction to Graph Theory.) Let G be a bipartite graph with vertex sets X and Y, so that there are no edges between vertices in X, and no edges between vertices of Y. Suppose that every vertex in X has a valency at least equal to the integer t and suppose that the Hall condition is satisfied. Show that the number of complete matchings in G is at least equal to $t!$ if $|X| \geq t$, and is at least equal to $t!/(t - |X|)!$ if $|X| < t$.