Matematiska Institutionen
KTH

A typical exam to the course Discrete Mathematics, SF2736, fall 12.

Observe:

1. Nothing else than pencils, rubbers, rulers and papers may be used.
2. Bonus points from the homeworks will be added to the sum of points on part I.
3. Grade limits: 13-14 points will give Fx; 15-17 points will give E; 18-21 points will give D; 22-27 points will give C; 28-31 points will give B; 32-36 points will give A.

Part I

1. (3p) The 1-error-correcting code C has the (parity) check-matrix

$$H = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

   (a) Find the number of words of C.
   (b) Find two words of C.
   (c) Find the number of words that the code cannot correct.

2. (a) (1p) Find all solutions to the equation $14x + 7 = 26$ in the ring Z_{37}.
   (b) (2p) Find $2^{200} (\mathrm{mod}\ 37)$.

3. (3p) Let G be a finite graph with at least three edges and assume that the valency (degree) of every vertex is an even non-zero integer. Explain why there must exist at least one cycle in G.

4. Answer, together with an explanation, the following questions:
   (a) (1p) Can you find any odd permutation the order of which is even?
   (b) (1p) Can you find any even permutation the order of which is odd?
   (c) (1p) Can you find any odd permutation the order of which is odd?
   (d) (1p) Can you find any even permutation the order of which is even?

5. Find the number of solutions to the Diophantine equation

$$x_1 + x_2 + x_3 + x_4 \leq 15,$$

   where it is required that $0 \leq x_1 \leq 4$, $-2 \leq x_2 \leq 3$, $2 \leq x_3 \leq 8$ and $x_4 \geq 0$.

Part II

6. (a) (1p) Prove that $\gcd(a, b)$ divides $\mathrm{lcm}(a, b)$ for all integers a and b.
   (b) (2p) Find the number of sets $\{x, y\}$ such that x and y are non negative integers and
$$\gcd(x, y) = 60 \qquad \text{and} \qquad \mathrm{lcm}(x, y) = 12600.$$

7. (4p) Let $\mathcal{S}_4$ denote the symmetrical group on four elements, that is, the group that consists of all permutations of the elements in the set $\{1, 2, 3, 4\}$. Find the smallest subgroup H of $\mathcal{S}_4$ that contains the permutations (1 2) and (2 3 4).

8. (4p) Find the number of surjective maps f from the set $\{1, 2, 3, \ldots, 10\}$ to $\{1, 2, 3, \ldots, 6\}$ such that the elements $f(1)$, $f(2)$ and $f(3)$ are distinct.

Part III

9. Let G be a k-regular bipartite graph G with the set of vertices X and Y, (so there are no edges between vertices in X and no edges between vertices in Y).
   (a) (1p) Show that the number of vertices in X is equal to the number of vertices in Y.
   (b) (1p) Show that if $k = 2$ then there will exist an edge-coloring of G in two colors.
   (c) (3p) Show that to every non-negative integer p less than k there is a coloring of the edges in G in the colors black and white such that at each vertex there will be p white edges and $k - p$ black edges.

10. (5p) Let $\varphi(n)$ denote the number of integers $1 \leq m \leq n$ that are relatively prime to n. Show that for all positive integers n, and all prime numbers p, it is true that n divides $\varphi(p^n - 1)$.