Matematiska Institutionen
KTH

**Exam to Discrete Mathematics, SF2736, December 14, 2012, 08.00–13.00.**

**Examiner:** Olof Heden

**Observe:**

1. You are not allowed to use anything else than pencils, rubber, rulers and papers at this exam.
2. To get the maximum number of points on a problem it is not sufficient to just give an answer, you must also provide explanations.
3. Bonus points from the homeworks will be added to the sum of the points on part I.
4. Grade limits: 13-14 points will give an Fx; 15-17 points will give an E; 18-21 points will give a D; 22-27 points will give a C; 28-31 points will give a B; 32-37 points will give an A.

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# Part I

1. (3p) Solve, by using the technique with generating functions, the recursion
$$a_0 = 2, \quad a_1 = 2, \quad \text{and,} \quad a_n = 2a_{n-1} + 8a_{n-2}, \quad \text{for} \quad n = 2, 3, \ldots.$$
2. (a) (1.5p) Find $5^{255} \pmod{127}$.
   (b) (1.5p) Find $5^{255} \pmod{129}$.
3. (3p) There are four classes in a school, each consisting of 15 children. A committee consisting of 12 children shall be chosen. In how many ways can this be done if it is required that the committee must have at least one child from each class.
4. Let $G$ denote the group which is the following direct product of the groups $(Z_3, +)$ and $(Z_2, +)$:
$$G = (Z_3, +) \times (Z_2, +) \times (Z_2, +) \times (Z_2, +).$$
   (a) (1.5p) Find one subgroup of size six to $G$.
   (b) (1.5p) Find the number of distinct subgroups of size six to $G$.
5. Let $G$ be a graph, with no loops and no multiple edges, with 1024 edges and 1024 vertices. Answer the following two question together with a short explanation.
   (a) (1p) If $G$ is connected then $G$ has at least one cycle. Why?
   (b) (2p) If $G$ consists of two components, which are then the possibilities for the number of cycles in $G$?

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# Part II

6. (3p) Are there any permutations $\varphi$, $\psi$ and $\delta$ in the symmetrical group $\mathcal{S}_{12}$, the group consisting of all permutations on the set $\{1, 2, \ldots, 12\}$, such that

$$\varphi^4 = (1\ 2)(3\ 4)(5\ 6), \qquad \psi^5 = (1\ 2)(3\ 4)(5\ 6), \quad \text{and} \quad \delta^6 = (1\ 2)(3\ 4)(5\ 6).$$

7. (4p) Find the number of surjective maps $f$ from the set $A = \{1, 2, \ldots, 9\}$ to the set $\{1, 2, \ldots, 5\}$ such that

$$|\{x \in A \mid f(x) = f(1)\}| = |\{x \in A \mid f(x) = f(2)\}|.$$

Besides explanations, your answer to this question must be given as an integer.

8. Let $C$ denote the set of words $(c_1, c_2, \ldots, c_{11})$ in $Z_2^{11} = Z_2 \times Z_2 \times \cdots \times Z_2$ such that

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{11} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

The code $C$ is an 1-error-correcting code (you do not need to prove this fact!).

(a) (1p) Correct the word $(0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0)$.

(b) (1p) Find a word that cannot be corrected.

(c) (1p) How many words cannot be corrected?

(d) (2p) Let $\bar{c}^T$ denote the transpose of a matrix $\bar{c}$. If the first column in the $4 \times 11$-matrix above is substituted by the column $[0\ 0\ 0\ 0]^T$, $[0\ 0\ 1\ 1]^T$, or $[1\ 0\ 1\ 1]$, then the matrix equation above gives, instead of $C$, three other codes $C_1$, $C_2$ and $C_3$, respectively. Are any of these three codes an 1-error-correcting code?

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# Part III

9. (5p) Let $G$ be a bipartite graph with the two sets of vertices $X$ and $Y$ of the same size. (No edge between any two vertices in $X$ and similarly for $Y$.) For any subset $A$ of $X$ and any subset $B$ of $Y$ let

$$R(A) = \{y \in Y \mid y \text{ is neighbor to at least one } x \in A\},$$

$$L(B) = \{x \in X \mid x \text{ is neighbor to at least one } y \in B\}.$$

Show that there is a subset $A$ of $X$ such that $|A| > |R(A)|$ if and only if there is at least one subset $B$ of $Y$ such that $|B| > |L(B)|$.

10. (5p) Let $a$, $b$ and $d$ be elements in a ring $Z_n$. Find the number of solutions $x$ and $y$ in $Z_n$ to the equation

$$ax + by = d.$$

(Partial solutions to this problem will give one or more points, for example you will get 1p if you solve the problem in the case $\gcd(a, n) = 1$.)