Matematiska Institutionen
KTH

Exam to the course Discrete Mathematics, SF2736, 08.00 to 13.00 on June 7, 2011.

Observe:

1. Nothing else than pencils, rubber, rulers and papers may be used.
2. Bonus points from the homeworks will be added to the sum of points on part I.
3. Grade limits: 13-14 points will give Fx; 15-17 points will give E; 18-21 points will give D; 22-27 points will give C; 28-31 points will give B; 32-36 points will give A.

Part I

1. (3p) Find the least positive remainder when 7^{1024} is divided by 31.
2. (3p) Draw three graphs G_1, G_2 and G_3, each with 12 vertices and 18 edges, with the following properties:

   (i) G_1 have an Euler circuit but no Hamiltonian cycle.

   (ii) G_2 have an Hamiltonian cycle but no Euler circuit

   (iii) G_3 have neither an Eulerian circuit nor an Hamiltonian cycle.
3. (3p) In how many ways can the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ be partitioned into three subsets such that the elements 1, 2 and 3 will be placed in different sets?
4. (3p) Let G denote the group $(Z_{24},\ +\ )$. Find cosets S_1 and S_2 of two distinct non trivial subgroups H_1 and H_2, respectively, of G, with the property that S_1 and S_2 have exactly two elements in common of which one is the element 7.
5. (a) (1p) Let A denote a set of positive integers. Give a suitable definition of the concept greatest common divisor, below denoted $\gcd(A)$, of the numbers in the set A.

   (b) (2p) Show that for every non empty subset B of A it is true that $\gcd(A)$ divides $\gcd(B)$.

Part II

6. (3p) Find the number of integers n in the interval $1 \leq n \leq 1320$ such that n is not divisible by 10, 11 or 12.

7. (3p) Find an 1-error correcting linear code C with as many words as possible and such that

   (i) the word 10101011 belongs to C.

   (ii) the word 00111100 cannot be corrected.

   (iii) the word 11000011 can be corrected.

8. (a) (2p) Find a non abelian (non commutative) group G with identity e with subgroups H_1, H_2, ..., H_k, where $k > 1$, of G such that $H_i \cap H_j = \{e\}$ for $i \neq j$, and such that
   $$H_1 \cup H_2 \cup \ldots \cup H_k = G \, .$$

   (b) (2p) Assume that G is a group with the property that every element, except the identity, has the same order p, where p is a prime number. Show that G has subgroups H_1, H_2, ..., H_k, $k > 1$, that partition the set of non identity elements of the group G, in the same way as the subgroups in problem 8 (a) do.

   (c) (1p) Let $p = 5$. Give an explicit example that demonstrates the facts in subproblem 8 (b).

Part III

9. Let $\mathcal{S}_n$ denote the set of all permutations of the elements in the set $\{1, 2, \ldots, n\}$. Two permutations α and β are said to commute if $\alpha\beta = \beta\alpha$. Below permutations are described by their cycle notation.

   (a) (1p) Find all elements in $\mathcal{S}_3$ that commute with the permutation $\alpha = (1\ 2\ 3)$.

   (b) (1p) Find five elements in $\mathcal{S}_5$ that commute with the permutation $\alpha = (1\ 2\ 3\ 4\ 5)$.

   (c) (3p) For every positive integer n, find all elements in $\mathcal{S}_n$ that commute with the permutation $\alpha = (1\ 2\ \ldots\ n)$.

10. We consider bipartite graphs with vertices in the sets X and Y with no edges between vertices in X and no edges between vertices in Y.

    (a) (2p) Show that if $|X| \leq 10$, if the vertices in the sets X have a degree at least equal to 4, and if the vertices of Y have a degree less than or equal to 5, then there always exists a matching of size 8 in the bipartite graph.

    (b) (3p) Give and prove a theorem that generalizes the above situation, and from which the result in problem 10 (a) follows.

    Note You can get 5p on problem 10 by first solving the (b)-problem and then showing how the result in (a) follows from the solution of problem (b).