Matematiska Institutionen
KTH

A typical exam to the course Discrete Mathematics, SF2736.

Observe:

1. Nothing else than pencils, rubber, rulers and papers may be used.
2. Bonus points from the homeworks will be added to the sum of points on part I.
3. Grade limits: 13-14 points will give Fx; 15-17 points will give E; 18-21 points will give D; 22-27 points will give C; 28-31 points will give B; 32-36 points will give A.

Part I

1. (3p) Find a and b such that
$$\binom{a}{b} = \binom{n-1}{k-1} + \binom{n-1}{k} + \binom{n}{n-k-1}$$
2. (3p) The 1-error correcting code C has the (parity) check matrix
$$H = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$
   (a) Find the number of words of C.
   (b) Find two words of C.
   (c) Find the number of words that the code cannot correct.
3. (3p) Find all solutions to the equation $14x + 7 = 26$ in the ring Z_{37}.
4. (3p) Let G be a finite graph with at least three edges and assume that the valency (degree) of every vertex is an even non zero number. Explain why there must exist at least one cycle in G.
5. Answer the following questions.
   (a) (1p) Do there exist odd permutations the order of which are even?
   (b) (1p) Do there exist even permutations the order of which are odd?
   (c) (1p) Do there exist odd permutations the order of which are odd?
   (d) (1p) Do there exist even permutations the order of which are even?

Part II

6. (a) (1p) Prove that $\gcd(a,b)$ divides $\operatorname{lcm}(a,b)$ for all integers a and b.

   (b) (2p) Find the number of sets $\{x,y\}$ such that x and y are non negative integers and

$$\gcd(x,y) = 60 \qquad \text{and} \qquad \operatorname{lcm}(x,y) = 12600.$$

7. (3p) Find the number of surjective maps f from the set $\{1,2,3,\ldots,10\}$ to the set $\{1,2,3,\ldots,6\}$ with the property that the elements $f(1)$, $f(2)$ and $f(3)$ are distinct elements.

8. (a) (1p) Consider the group $G = (Z_{20},+)$. Find two distinct subgroups H and K to G with cosets $a+H$ and $b+K$, where $a \notin H$ and $b \notin K$, such that

$$|(a+H)\cap(b+K)| = 2.$$

   (b) (3p) Consider any group G. Is it true in general that if $a+H$ and $b+K$ are cosets to subgroups H and K of G then the number of elements in $(a+H)\cap(b+K)$ either are equal to zero or divides the number of element of G, i.e., is it in general true that

$$|(a+H)\cap(b+K)| = m \quad \text{and} \quad |G| = n \quad \Longrightarrow \quad m = 0 \quad \text{or} \quad m \mid n.$$

Part III

9. Let G be a k-regular bipartite graph G with the set of vertices X and Y, so there are edges just from vertices in X to vertices in Y.

   (a) (1p) Show that the number of vertices of X is equal to the number of vertices of Y.

   (b) (1p) Show that if $k = 2$ then there will exist an edge-coloring of G in two colors.

   (c) (3p) Show that if p is any non negative integer less than k, then there will exist a coloring of the edges of G in the colors black and white such that at each vertex there will be p white edges and $k-p$ black edges.

10. (5p) Let $\varphi(n)$ denote the number of integers $1 \leq m \leq n$ that are relatively prime to n. Show that for all positive integers n, and all prime numbers p, it is true that n divides $\varphi(p^n - 1)$.