KTH Teknikvetenskap

SF2729 Groups and Rings
Final Exam
Wednesday, August 19, 2010

Time: 08.00-12.00
Allowed aids: none
Examiner: Mats Boij

This final exam consists of two parts; Part I (groups part) and Part II (rings part). The final credit for Part I will be based on the maximum of the results on the midterm exam and Part I in the final exam.

Each problem can give up to 6 points. In the first problem of each part, you are guaranteed a minimum given by the result of the corresponding homework assignment. If you have at least 2 points from HW1, you cannot get anything from Part a) of Problem 1 of Part I, if you have at least 4 points from HW1 you cannot get anything from Part a) or Part b) of Problem 1 of Part I. Similarly for HW2 and Problem 1 of Part II.

The minimum requirements for the various grades are according to the following table:

Grade	**A**	**B**	**C**	**D**	**E**
Total credit	30	27	24	21	18
From Part I	13	12	11	9	8
From Part II	13	12	11	9	8

Present your solutions to the problems in a way such that arguments and calculations are easy to follow. Provide detailed arguments to your answers. An answer without explanation will give no points.

Part I - Groups

(1) (a) Give an example of a binary operation on $S = \{1, 2, 3\}$ which is commutative with a unit, but which fails to be associative. **(2)**
(b) Show that any finite cyclic group has exactly one subgroup of any order dividing the order of the group. **(2)**
(c) For all integers $n \geq 2$, compute the center of the dihedral group, D_{2n}, i.e. the group of symmetries of a regular n-gon. **(2)**

(2) (a) Show that the center of any group is a normal subgroup and deduce that any simple group has a trivial center. **(2)**
(b) Let $\Phi : G \longrightarrow H$ be a group homomorphism and let K be a normal subgroup of H. Show that $\Phi^{-1}(K) = \{a \in G | \Phi(a) \in K\}$ is a normal subgroup of G. **(2)**
(c) Show that in the situation described in (2b) we get an induced homomorphism

$$\tilde{\Phi} : G/\Phi^{-1}(K) \longrightarrow H/K.$$

(2)

(3) A group G which acts on a set X is said to act *freely* if all stabilizers are trivial.
(a) Show that any group acts freely on itself by left multiplication. **(1)**
(b) Show that if a finite group G acts freely on a non-empty set X, then $|X| \geq |G|$. **(2)**
(c) Show that any free action of a group G can be identified with the action of the group on a union of copies G where G acts by left multiplication on each copy of G. **(3)**

Part II - Rings

(1) Consider the function $\phi : \mathbb{Z}[x] \rightarrow \mathbb{Z}_8$ defined by $f(x) \mapsto [f(3)]_8$.
(a) Show that ϕ is a ring homomorphism. **(1)**
(b) Show that $ker(\phi)$ is not a prime ideal. **(2)**
(c) Show that $ker(\phi)$ is finitely generated and find a finite set of generators. **(3)**

(2) Consider the field with q elements, $\mathbb{F}_q$, and the polynomial $f(x) = x^2 + 1 \in \mathbb{F}_q[x]$. Let $K = \mathbb{F}_q[x]/(f(x))$.
(a) Compute the number of elements in K. **(2)**
(b) Determine all integers q for which K is a field. **(4)**

(3) Let A be a commutative ring with unity. An element $a \in A$ is said to be *nilpotent* if $a^k = 0$ for some k. Let $N(A)$ be the set of all nilpotent elements of A.
(a) Show that $N(A)$ is an ideal. **(2)**
(b) Show that all $N(A)$ is contained in every prime ideal of A. **(1)**
(c) Show that $N(A)$ is the intersection of all prime ideals of A. **(3)**