KTH, Matematik

Svar (med reservation för felräkningar) till Kapitel 8.3-8.6

8.3.1 (a) $\mathrm{Ker}(T) = \{0\}$; T is injective. (b) $\mathrm{Ker}(T) = \{t(-3/2, 1) \mid t \in \mathbb{R}\}$; T is not injective. (c) $\mathrm{Ker}(T) = \{0\}$; T is injective. (d) $\mathrm{Ker}(T) = \{0\}$; T is injective. (e) $\mathrm{Ker}(T) = \{t(1, 1) \mid t \in \mathbb{R}\}$; T is not injective. (f) $\mathrm{Ker}(T) = \{t(0, 1, -1) \mid t \in \mathbb{R}\}$; T is not injective.

8.3.3 (a) T has no inverse,

$$(b)\ T^{-1}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{8}x_1 + \frac{1}{8}x_2 - \frac{3}{4}x_3 \\ \frac{1}{8}x_1 + \frac{1}{8}x_2 + \frac{1}{4}x_3 \\ -\frac{3}{8}x_1 + \frac{5}{8}x_2 + \frac{1}{4}x_3 \end{pmatrix},\ \text{(c) } T^{-1}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ -\frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 \end{pmatrix},$$

$$\text{(d) } T^{-1}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 + 3x_2 - x_3 \\ -2x_1 - 2x_2 + x_3 \\ -4x_1 - 5x_2 + 2x_3 \end{pmatrix}.$$

8.3.10 a) no, b) yes, c) yes.

8.3.16 Let V, W be finite dimensional vector spaces. Prove that if $\dim(W) < \dim(V)$, then there exists no injective linear map $T : V \to W$.
Proof: Suppose that there exists an injective linear map $T : V \to W$. Then we have

$$\dim(V) = \underbrace{\dim\mathrm{Ker}(T)}_{=0} + \underbrace{\dim\mathrm{Im}(T)}_{\leq \dim(W)} \leq \dim(W),$$

which contradicts $\dim(W) < \dim(V)$.

8.4.9 a) $(T(v_1))_B = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $T(v_2)_B = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ b) $(T(v_1)) = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$, $T(v_2) = \begin{pmatrix} -2 \\ 29 \end{pmatrix}$, c) $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{18}{7} & \frac{1}{7} \\ -\frac{107}{7} & \frac{24}{7} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, d) $\begin{bmatrix} \frac{19}{7} \\ -\frac{83}{7} \end{bmatrix}$.

8.4.15 Let $T : V \to V$ be a contraction or a dilation of V. Prove that the matrix of T with respect to any basis of V is a positive scalar multiple of the identity matrix.
Proof: Since T is a contraction or a dilation, there exists $\lambda > 0$ such that $T(v) = \lambda v$ for every $v \in V$. ($0 < \lambda \leq 1$ if T is a contraction and $1 \leq \lambda$ if T is a dilation.) In particular, for any basis $B = \{b_1, ..., b_n\}$ of V, we also have $T(b_i) = \lambda b_i$, for every i. Thus, the coordinates of $T(b_i)$ with respect to the basis B are

$$i \begin{pmatrix} 0 \\ \vdots \\ \lambda \\ \vdots \\ 0 \end{pmatrix}$$

(a λ in the i-th row and 0's otherwise). The matrix of T with respect to the basis

V is hence equal to

$$\begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda \end{pmatrix} = \lambda I_n.$$

8.4.16

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

8.5.5 $(T)_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $(T)_{B'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

8.5.6

$$\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

8.5.7

$$\begin{pmatrix} 9/4 & 9/2 \\ 3/2 & 1 \end{pmatrix}$$

8.5.16 (a) Prove that if A, B are similar, then A^2 and B^2 are similar, and more generally, A^k and B^k are similar for any positive integer k.

Proof: Since A and B are similar, there exists an invertible matrix P such that $A = PBP^{-1}$. Thus, we have

$$A^2 = A \cdot A = (PBP^{-1})(PBP^{-1}) = PB\underbrace{P^{-1}P}_{\text{Id}}BP^{-1} = PBBP^{-1} = PB^2P^{-1},$$

so that A^2 and B^2 are similar. More generally, let k be any positive integer. Then

$$\begin{aligned} A^k &= (PBP^{-1})^k = (PBP^{-1})(PBP^{-1})(PBP^{-1}) \cdot \ldots \cdot (PBP^{-1}) \\ &= PB\underbrace{P^{-1}P}_{\text{Id}}B\underbrace{P^{-1}P}_{\text{Id}}BP^{-1} \cdot \ldots \cdot PBP^{-1} = PB^kP^{-1}, \end{aligned}$$

so that A^k and B^k are similar.

(b) However, it is not true that if A^2 and B^2 are similar, than A and B are similar. For example, take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then A^2 and B^2 are similar since they are equal, but A and B cannot be similar, since $PBP^{-1} = B \neq A$ for any invertible matrix P.

8.6.5 (a) No, not surjective, (b) Yes, (c) No, not injective, (d) No, not injective.

8.6.7 Prove that there exists a surjective linear map $V \to W$ if and only if $\dim(V) \geq \dim(W)$.

Proof: Suppose that there exists a surjective linear map $f : V \to W$. Then

$$\dim(V) = \underbrace{\dim\mathrm{Ker}(f)}_{\geq 0} + \underbrace{\dim\mathrm{Im}(f)}_{=\dim(W)} \leq \dim(W).$$

Conversely, suppose that $n = \dim(V) \geq \dim(W) = m$. Let $\{v_1, ..., v_n\}$ be a basis of V and $\{w_1, ..., w_m\}$ be a basis of W. Define a linear map $f : V \to W$ by

$$f(v_i) = \begin{cases} w_i & \text{if } i \leq m, \\ 0 & \text{if. } i > m. \end{cases}$$

The map f is surjective, since every vector of the basis $\{w_1, ..., w_m\}$ of W lies in its image.

8.6.8

$$a) \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} \mapsto (a, b, c, d, e, f). \text{ b) } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c, d).$$

$$\text{c) } 0 + at + bt^2 + ct^3 \mapsto (a, b, c). \text{ d) } a + b \sin x + c \cos x \mapsto (a, b, c).$$