LINEAR ALGEBRA BACKGROUND FOR MATHEMATICAL SYSTEMS THEORY.

1. Introduction

In this note we summarize some linear algebra results that will be used when investigating reachability and observability of linear systems. The basic problem under consideration is the fundamental solvability conditions for linear equation systems.

Warning: The presentation is compact and dense. The main point is Figure 1. If you understand the idea behind this figure then you will also be able to understand the idea behind the proof of reachability in Lindquist and Sand.

Consider the linear equation system

$$Ax = b \tag{1.1}$$

We will address the following questions

(a) When is there a solution?
(b) Is the solution unique? Which solution do we pick otherwise?
(c) How do we construct the solution?

As an example consider the simple system

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \beta \end{bmatrix} \tag{1.2}$$

It is clear that

(a) There exist solutions if and only if $\beta = 1$.
(b) If $\beta = 1$ then any solution must satisfy $x_1 + x_2 = 1$. This equation defines a line, which means that there are infinitely many solutions to (1.2) if $\beta = 1$ and otherwise no solutions at all. Among all these solutions it is natural to pick the minimal length solution, i.e. the solution to the optimization problem

$$\min x_1^2 + x_2^2 \quad \text{subj. to} \quad x_1 + x_2 = 1$$

(c) In this simple problem we reduce the optimization problem to

$$\min x_1^2 + (1 - x_1)^2 = \min 2x_1^2 - 2x_1 + 1$$

which has the solution $x_1 = 1/2$, which gives $(x_1, x_2) = (0.5, 0, 5)$.

We will formalize the above discussion by using the fundamental theorem of linear algebra. For a comprehensive discussion of these topics we refer to

- K. Svanberg. Linjär algebra för optimerare.
- G. Strang. Linear algebra and it's applications. (The key pages are attached in Trygger's OH collection).

This note borrows material from an exercise note by G. Bortolin.

2. Some key linear algebra definitions

We here summarize some basic concepts and definitions in linear algebra that will be used.

- The set $\mathbb{R}^n$ is the set of all column vectors with n real valued components. Any element in $x \in \mathbb{R}^n$ will be represented as a column vector
$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$
- $\mathbb{R}^n$ is an example of a (finite dimensional) vector space. The formal definition of a linear vector space is given next.

  Definition 2.1 (Linear vector space)**.** A (real) linear vector space is a set $\mathcal{V}$ of elements that satisfies the following rules of operation. If $x, y, z \in \mathcal{V}$, $a, b \in \mathbb{R}$, and 0 is the zero vector then
  (i) $(x + y) + z = x + (y + z)$
  (ii) $0 + x = x$
  (iii) $x + (-x) = 0$
  (iv) $x + y = y + x$
  (v) $a(x + y) = ax + ay$
  (vi) $(a + b)x = ax + by$
  (vii) $(ab)x = a(bx)$
  (viii) $1 \cdot x = x$

  In $\mathbb{R}^n$, 0 is the vector with all components equal to zero.

  You probably use this set rules all the time without really reflecting on its value. The importance of the definition is that it is not only $\mathbb{R}^n$ that satisfies the above rules. There are other important sets of elements that satisfy the same rules of operation and therefore defines a linear vector space. One example is linear subspaces of $\mathbb{R}^n$ that will be discussed next.
- A subset $\mathcal{V} \subset \mathbb{R}^n$ is a linear subspace if $\forall v_1, v_2 \in \mathcal{V}$ and $\forall \alpha_1, \alpha_2 \subset \mathbb{R}$ $\alpha_1 v_1 + \alpha_2 v_2 \in \mathcal{V}$. A linear subspace $\mathcal{V} \subset \mathbb{R}^n$ is a vector space since it satisfies the operations in Definition 2.1. Indeed, since the subspace is closed under linear combinations, i.e. $\alpha_1 v_1 + \alpha_2 v_2 \in \mathcal{V}$, for all $\alpha_1, \alpha_2 \in \mathbb{R}$ and $v_1, v_2 \in \mathcal{V}$, it follows that the rules are inherited from $\mathbb{R}^n$.
- A subspace $\mathcal{V} \subset \mathbb{R}^n$ is said to be *spanned* by the set of vectors $v_1, \ldots, v_r \in \mathcal{V}$ if every $v \in \mathcal{V}$ can be written as a linear combination of the v_k, i.e., there exists $\alpha_k \in \mathbb{R}$ such that $v = \sum_{k=1}^r \alpha_k v_k$. The above can in more compact notation be written
$$\mathcal{V} = \text{span}\{v_1, \ldots, v_r\} := \left\{ \sum_{k=1}^r \alpha_k v_k : \alpha_k \in \mathbb{R} \right\}$$

  The set of vectors $v_1, \ldots, v_r$ is called *linearly independent* if $\sum_{k=1}^r \alpha_k v_k = 0$ implies $\alpha_k = 0$, $k = 1, \ldots, r$. A linearly independent set of vectors that spans $\mathcal{V}$ is called a *basis* for $\mathcal{V}$.

  If $\mathcal{V} = \text{span}\{v_1, \ldots, v_r\}$ where the vectors v_k, $k = 1, \ldots, r$ are linearly independent (i.e. a basis for $\mathcal{V}$) then we say that $\mathcal{V}$ has dimension r, which is denoted $\dim \mathcal{V} = r$.

- A vector space $\mathcal{V}$ is a *direct sum* of two subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$, which is written $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$, if every vector $v \in \mathcal{V}$ uniquely can be decomposed as

$$v = v_1 + v_2, \quad v_1 \in \mathcal{V}_1,\ v_2 \in \mathcal{V}_2$$

- The vector spaces used in the course are always equipped with a so-called inner product (scalar product). For $\mathbb{R}^n$ the scalar product and its associated norm are defined as

$$x^T y = \sum_{k=1}^{n} x_k y_k$$

$$\|x\| = \sqrt{x^T x} = \sqrt{\sum_{k=1}^{n} x_k^2}$$

More generally we have the following definition

Definition 2.2 (Inner product)**.** An inner product in a (real) vector space $\mathcal{V}$ is a mapping $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ such that
(*i*) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$
(*ii*) $\langle x_1, x_2 \rangle = \langle x_2, x_1 \rangle$
(*iii*) $\langle x_1, a x_2 + b x_3 \rangle = a \langle x_1, x_2 \rangle + b \langle x_1, x_3 \rangle$
The norm associated with the inner product is defined as

$$\|x\| = \sqrt{\langle x, x \rangle}$$

and satisfies the properties
(*a*) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$
(*b*) $\|\alpha x\| = |\alpha| \|x\|, \qquad \forall \alpha \in \mathbb{R}$.
(*c*) $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$ (triangle inequality)

Definition 2.3. An inner product space is a vector space $\mathcal{V}$ equipped with an inner product.

When investigating reachability and observability of a linear system we use the inner product space

$$\mathrm{L}_2^m[t_0, t_1] = \{u : [t_0, t_1] \to \mathbb{R}^m \ s.t. \ \int_{t_0}^{t_1} \|u(t)\|^2 dt < \infty\}$$

The inner product and norm on this vector space are defined as

$$\langle u, v \rangle_{\mathrm{L}_2} = \int_{t_0}^{t_1} u(t)^T v(t) dt$$

$$\|u\|_{\mathrm{L}_2} = \sqrt{\langle u, u \rangle_{\mathrm{L}_2}} = \left(\int_{t_0}^{t_1} \|u(t)\|^2 dt \right)^{1/2}$$

Now that the inner product is defined we can introduce the important concept of orthogonal complements and orthogonal direct sum decompositions.

- Let $\mathcal{V}$ be a vector space and $\mathcal{W} \subset \mathcal{V}$ a subspace. The *orthogonal complement* of $\mathcal{W}$, denoted $\mathcal{W}^\perp$, is defined as

$$\mathcal{W}^\perp = \{v \in \mathcal{V} : \langle v, w \rangle = 0 \ \ \forall w \in \mathcal{W}\}$$

The orthogonal complement provides a particularly nice direct sum decomposition $\mathcal{V} = \mathcal{W} \oplus \mathcal{W}^\perp$.

The orthogonal complement satisfies $(\mathcal{V}^\perp)^\perp = \overline{\mathcal{V}}$. Here $\overline{\mathcal{V}}$ denotes the closure of $\mathcal{V}$. In all cases considered in this course we can use the simplified rule $(\mathcal{V}^\perp)^\perp = \mathcal{V}$.

- The notation $\mathcal{V}_1 \perp \mathcal{V}_2$ means that the vectors in $\mathcal{V}_1$ and $\mathcal{V}_2$ are mutually orthogonal, i.e. $\langle v_1, v_2 \rangle = 0$ for any $v_1 \in \mathcal{V}_1$ and $v_2 \in \mathcal{V}_2$.

Example 2.4. If

$$\mathcal{V}_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}, \qquad \mathcal{V}_2 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{V}_3 = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

then $\mathbb{R}^3 = \mathcal{V}_1 \oplus \mathcal{V}_2$ is a direct sum decomposition and $\mathbb{R}^3 = \mathcal{V}_1 \oplus \mathcal{V}_3$ is an orthogonal direct sum decomposition, i.e. $\mathcal{V}_3 = \mathcal{V}_1^\perp$.

3. The fundamental theorem of linear algebra

Let us consider an $m \times n$ matrix A. Given A, we can define four *fundamental subspaces*[1]:

1. The column space of A, defined as $\operatorname{Im} A = \{Ax : x \in \mathbb{R}^n\} \subset \mathbb{R}^m$
2. The null space of A, defined as $\operatorname{Ker} A = \{x \in \mathbb{R}^n : Ax = 0\} \subset \mathbb{R}^n$.
3. The row space of A, which is the column space of A^T, defined as $\operatorname{Im} A^T = \{A^T y : y \in \mathbb{R}^m\} \subset \mathbb{R}^n$.
4. The left null space of A, which is the null space of A^T, defined as $\operatorname{Ker} A^T = \{y \in \mathbb{R}^m : y^T A = 0\} \subset \mathbb{R}^m$.

Definition 3.1. The *rank* of A is defined as $\operatorname{rank}(A) = \dim \operatorname{Im} A$.

There is an important relation between these four subspaces described in the following theorem:

The fundamental theorem of linear algebra: *Suppose A is a $m \times n$ matrix and has rank r. Then, one has the following decomposition:*

$$\mathbb{R}^n = \operatorname{Im} A^T \oplus \operatorname{Ker} A \quad \text{and} \quad \mathbb{R}^m = \operatorname{Im} A \oplus \operatorname{Ker} A^T \tag{3.1}$$

where the dimensions are:

$$\begin{aligned} \dim \operatorname{Im} A &= r \\ \dim \operatorname{Im} A^T &= r \\ \dim \operatorname{Ker} A &= n - r \\ \dim \operatorname{Ker} A^T &= m - r \end{aligned}$$

and most importantly the decompositions in (3.1) are orthogonal, i.e.

$$(\operatorname{Im} A)^\perp = \operatorname{Ker} A^T$$

$$(\operatorname{Im} A^T)^\perp = \operatorname{Ker} A$$

What does the fundamental theorem of algebra tell us about the $m \times n$ matrix A? We have the following consequences:

1. It tells us the dimensions of the four subspaces. In particular $\dim \operatorname{Im} A = \dim \operatorname{Im} A^T$.

[1] A common alternative notation is $\mathcal{R}(A) := \operatorname{Im} A$ and $\mathcal{N}(A) := \operatorname{Ker} A$. Note that in the course, $\mathcal{R}$ denotes the reachability matrix. For a proof that the fundamental subspaces indeed are subspaces we refer to Svanberg, Linjär algebra för optimerare.

(2) The ambient space in which each subspace lives. That is:
  - $\operatorname{Im} A$ and $\operatorname{Ker} A^T$ are in $\mathbb{R}^m$.
  - $\operatorname{Im} A^T$ and $\operatorname{Ker} A$ are in $\mathbb{R}^n$

(3) The direct sum statement tells us that:
  - Every vector $b \in \mathbb{R}^m$ can be written as the sum of a vector $b_{col} \in \operatorname{Im} A$ and $b_{lnull} \in \operatorname{Ker} A^T$:

$$b = b_{col} + b_{lnull}$$

  - Every vector b in $\mathbb{R}^n$ can be written as the sum of a vector $b_{row} \in \operatorname{Im} A^T$ and $b_{null} \in \operatorname{Ker} A$:

$$b = b_{row} + b_{null}$$

(4) The orthogonality of the four subspaces:
  - Every vector in $\operatorname{Im} A$ is orthogonal to every vector in $\operatorname{Ker} A^T$.
  - Every vector in $\operatorname{Im} A^T$ is orthogonal to every vector in $\operatorname{Ker} A$.

3.1. **Minimum norm solutions to linear equation systems.** Let us again consider the linear equation:

$$Ax = b \tag{3.2}$$

We can now answer our previous questions.

(a) There is a solution to (3.2) if and only if $b \in \operatorname{Im} A$

(b) Supposing that $b \in \operatorname{Im} A$ we have two possibilities. Either $\operatorname{Ker} A = 0$ and the solution is unique or each solution of (3.2) can be uniquely decomposed into

$$x = x_{row} + x_{null} \quad \text{with } x_{row} \in \operatorname{Im}(A^T), x_{null} \in \operatorname{Ker}(A)$$

However we have that:

$$Ax = A(x_{row} + x_{null}) = Ax_{row} = b$$

and hence $x_{row} \in \operatorname{Im}(A^T)$ is the only part of x significant in generating b. Since x_{row} and x_{null} are orthogonal it follows that the x_{row} is the minimum norm solution, i.e. it solves the minimization problem

$$\min \|x\|^2 \quad \text{subj. to} \quad Ax = b$$

(c) We have three cases (assuming $b \in \operatorname{Im}(A)$)

  (i) $\operatorname{Ker} A = 0$ and $m = n$. Then A is invertible and the solution is obtained as $x = A^{-1}b$.

  (ii) If $\operatorname{Ker} A = 0$ but $m > n$ then we can use that $(\operatorname{Im} A)^\perp = \operatorname{Ker} A^T$, which implies that

$$Ax = b \qquad \Leftrightarrow \qquad A^T Ax = A^T b$$

Indeed, the implication in the right direction is obvious and for the left implication we use that $b, Ax \in \operatorname{Im}(A) = \operatorname{Ker}(A^T)^\perp$, which in turn implies that $Ax - b = 0$. We use $\operatorname{Ker} A^T A = \operatorname{Ker} A = 0$ (a formal proof can be found in Theorem 3.4.3 in Lindquist and Sand), which implies that $A^T A$ is invertible and the solution is obtained as $x = (A^T A)^{-1} A^T b$.

(*iii*) If $\operatorname{Ker} A \neq 0$ it follows from (*b*) that the minimum norm solution of the linear equation (3.2) is given by $x = x_{row} = A^T z$ where z is any solution of $AA^T z = b$. In particular, if AA^T is invertible the minimum norm solution is $x = A^T(AA^T)^{-1}b$. To see that this indeed is the minimum norm solution we let $x = A^T z + x_{null}$, where $x_{null} \in \operatorname{Ker} A$. We have

$$\begin{aligned}\|x\|^2 &= \|A^T z\|^2 + 2(A^T z)^T x_{null} + \|x_{null}\|^2 \\ &= \|A^T z\|^2 + 2z^T A x_{null} + \|x_{null}\|^2 \\ &= \|A^T z\|^2 + \|x_{null}\|^2 \geq \|A^T z\|^2\end{aligned}$$

with equality if and only if $x_{null} = 0$.

The above discussion leads to the following theorems. Note that a few gaps must be filled in order to have a complete proof, see Svanberg for more details.

Theorem 1: *If* $b \in \operatorname{Im} A$ *and* $\operatorname{Ker} A = 0$ *then the unique solution to* $Ax = b$ *can be obtained as* $x = (A^T A)^{-1} A^T b$.

Theorem 2: *We have* $\operatorname{Ker} A = \operatorname{Ker} A^T A$.

Theorem 3: *If* $b \in \operatorname{Im} A$ *but* $\operatorname{Ker} A \neq 0$ *then the unique minimum norm solution to* $Ax = b$ *can be constructed as* $x_{row} = A^T z$ *where* z *solves* $AA^T z = b$.

Theorem 4: *We have* $\operatorname{Im} A = \operatorname{Im} AA^T$.

Graphical interpretation to help memorize the last two results.

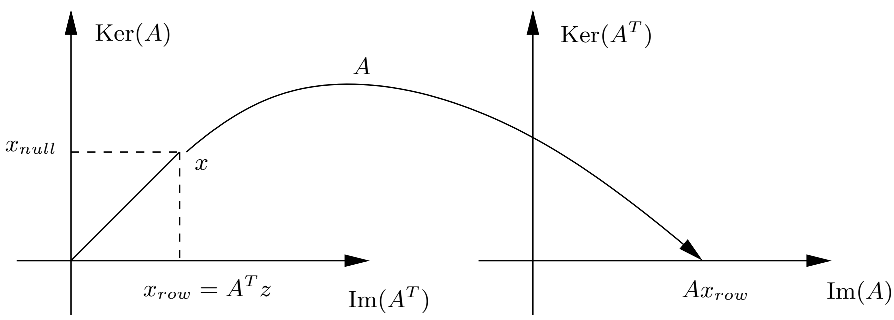

FIGURE 1. Illustration of the fundamental theorem of linear algebra.

Let us illustrate the Theorem 1 and Theorem 3 on three simple results.

Example 3.2. Consider again the system in (1.2) for the case when $\beta = 1$. We have $\operatorname{Ker} A = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ so we must use Theorem 3. We have

$$AA^T z = b \quad \Leftrightarrow \quad \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad z = \frac{1}{2} \begin{bmatrix} \alpha \\ 1-\alpha \end{bmatrix}$$

where $\alpha \in \mathbb{R}$ is arbitrary. We get

$$x = A^T z = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ 1-\alpha \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The next example shows a simple *under-determined system.*

Example 3.3. The equation system $Ax = b$ with $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $b = 1$ has the solution $x = A^T(AA^T)^{-1}b = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Our final example shows a simple *over-determined system*

Example 3.4. The equation system $Ax = b$ with $A = b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has the solution $x = (A^TA)^{-1}A^Tb = 1$.

In reachability analysis we will encounter infinite dimensional under-determined systems and in observability analysis we will encounter infinite dimensional over-determined systems.

How to determine the fundamental spaces in practice? Svanberg presents two methods to compute the fundamental subspaces.

(1) The first method uses the Gauss-Jordan elimination to factorize the matrix according to
$$A = PT$$
where P is an invertible matrix corresponding to elementary row operations on A and T is a matrix on stair case form. For computation we use $\operatorname{Ker} A = \operatorname{Ker} T$ and $\operatorname{Im} A = P\operatorname{Im} T$. Finally, $\operatorname{Im} A^T = (\operatorname{Ker} A)^{\perp}$ and $\operatorname{Ker} A^T = (\operatorname{Im} A)^{\perp}$.

(2) A more numerically oriented method is to use the singular value decomposition, see Svanberg.

We provide an example for the Gauss-Jordan method.

Example 3.5. Let
$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix}$$
Then $A = PT$ with
$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
We see directly that
$$\operatorname{Im} T = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \Rightarrow \quad \operatorname{Im} A = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$
For $\operatorname{Ker} A$ we have
$$\operatorname{Ker} A = \operatorname{Ker} T = \operatorname{Ker}\left(\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \right)$$
There are many ways to determine this nullspace. Svanberg provides a formula. For this simple example we use the definition and solve the equation system
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{cases} \alpha_3 = -\alpha_1 \\ \alpha_4 = -\alpha_2 - \alpha_3 = \alpha_1 - \alpha_2 \end{cases}$$

Hence, we have shown

$$\operatorname{Ker} A = \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ -\alpha_1 \\ \alpha_1 - \alpha_2 \end{bmatrix} : \alpha_k \in \mathbb{R},\ k = 1,2 \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

In the same way we determine $\operatorname{Ker} A^T = (\operatorname{Im} A)^\perp$ from the equation system

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{cases} \alpha_2 = -\alpha_1 \\ \alpha_3 = -\alpha_1 \end{cases}$$

This shows

$$\operatorname{Ker} A^T = \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \alpha_1 : \alpha_1 \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$$

and similarly we can obtain

$$\operatorname{Im} A^T = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

4. Generalization to Hilbert Space

The ideas behind the fundamental theorem of linear algebra can be generalized to Hilbert spaces. A Hilbert space is a special type of inner-product space and is usually denoted $\mathcal{H}$. In this note we concentrate on the two Hilbert spaces $\mathbb{R}^n$ and $\mathrm{L}_2^m[t_0, t_1]$, which have already been introduced. We are used to work with $\mathbb{R}^n$ but the function space $\mathrm{L}_2^m[t_0, t_1]$ is less familiar. One difference is that any basis for $\mathrm{L}_2^m[t_0, t_1]$ must have infinitely many elements. One possible basis representation would be the Fourier serious expansion.

Fortunately, for the reachability and observability problems the situation is easy and the results from the previous section are easy to generalize. The main difference from before is that the matrices are replaced by linear operators. The formal definition and main properties for our use is the following

Definition 4.1. A bounded linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a transformation between the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ that satisfies the following property. For all $v_1, v_2 \in \mathcal{H}_1$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ we have

$$A(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 A v_1 + \alpha_2 A v_2$$

The boundedness assumption means that there exists $c > 0$ such that

$$\|Av\|_{\mathcal{H}_2} \leq c\|v\|_{\mathcal{H}_1}, \qquad \forall v \in \mathcal{H}_1$$

where $\|\cdot\|_{\mathcal{H}_k}$, $k = 1, 2$ denotes the norm in Hilbert space $\mathcal{H}_k$. The boundedness means that A has finite amplification.

As a final definition we introduce the adjoint operator, which is a generalization of the matrix transpose

Definition 4.2. The adjoint of a bounded linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator $A^* : \mathcal{H}_2 \to \mathcal{H}_1$ defined by

$$\langle v, Aw \rangle_{\mathcal{H}_2} = \langle A^* v, w \rangle_{\mathcal{H}_1}, \qquad \forall v \in \mathcal{H}_2, w \in \mathcal{H}_1$$

Remark 4.3. There always exists a unique adjoint A^* to any bounded linear operator.

4.1. **Reachability Analysis.** In reachability analysis we consider the operator $L : \mathrm{L}_2^m[t_0, t_1] \to \mathbb{R}^n$ defined as

$$Lu = \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) u(\tau) d\tau$$

where $\Phi(t, \tau)$ is the transition matrix corresponding to the homogeneous system $\dot{x} = A(t)x$. The corresponding adjoint operator $L^* : \mathbb{R}^n \to \mathrm{L}_2^m[t_0, t_1]$

$$\begin{aligned} \langle d, Lu \rangle_{\mathbb{R}^n} &= d^T \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) u(\tau) d\tau \\ &= \int_{t_0}^{t_1} (B(\tau)^T \Phi(t_1, \tau)^T d)^T u(\tau) d\tau \\ &= \langle L^* d, u \rangle_{\mathrm{L}_2} \end{aligned}$$

This shows that

$$(L^* d)(t) = B(t)^T \Phi(t_1, t)^T d.$$

The reachability problem is from a mathematical point of view equivalent to the problem of finding $u \in \mathrm{L}_2^m[t_0, t_1]$ such that $Lu = d$. To understand this problem we use that the fundamental theorem of linear algebra generalizes also to the case with bounded linear operators in Hilbert space[2]. In particular, if we let

$$\begin{aligned} \mathrm{Im}\,(L) &= \{Lu : u \in \mathrm{L}_2^m[t_0, t_1]\} \subset \mathbb{R}^n \\ \mathrm{Ker}\,(L) &= \{u \in \mathrm{L}_2^m[t_0, t_1] : Lu = 0\} \subset \mathrm{L}_2^m[t_0, t_1] \\ \mathrm{Im}\,(L^*) &= \{L^* x : x \in \mathbb{R}^n\} \subset \mathrm{L}_2^m[t_0, t_1] \\ \mathrm{Ker}\,(L^*) &= \{x \in \mathbb{R}^n : L^* x = 0\} \subset \mathbb{R}^n \end{aligned}$$

then

$$\begin{aligned} \mathrm{L}_2^m[t_0, t_1] &= \mathrm{Im}\,(L^*) \oplus \mathrm{Ker}\,(L) & \qquad \mathbb{R}^n &= \mathrm{Im}\,(L) \oplus \mathrm{Ker}\,(L^*) \\ \mathrm{Im}\,(L^*) &\perp \mathrm{Ker}\,(L) & \mathrm{Im}\,(L) &\perp \mathrm{Ker}\,(L^*) \\ \dim \mathrm{Im}\,(L) &= \dim \mathrm{Im}\,(L^*) \end{aligned}$$

This leads to the same graphical illustration of the mapping L as for the mapping A in Figure 1.

This means that the answers to our three basic questions on existence of solution, uniqueness of solution, and construction algorithm for a solution to the equation system $Lu = d$ have the following answers

(a) there exists a solution iff $d \in \mathrm{Im}\, L = \mathrm{Im}\, LL^*$

[2]To learn more about this we refer to the book *Optimization in vector space*, by D. G. Luenberger. Note that the material in this book is beyond the scope of this course.

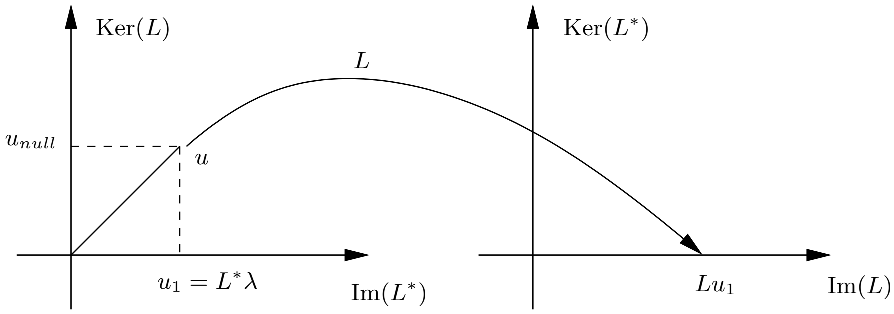

FIGURE 2. The main difference from Figure 1 is that the coordinate system in the left hand diagram represents the infinite dimensional space $L_2^m[t_0, t_1]$. The horizontal axis is still finite dimensional but the vertical axis is infinite dimensional.

(*b*) $\dim \operatorname{Ker} L \neq 0$ so there is not a unique solution. We therefore choose the minimum norm solution, i.e. the solution to the optimization problem

$$\min \|u\|_{L_2}^2 \quad \text{s.t.} \quad Lu = d \tag{4.1}$$

(*iii*) The solution to (4.1) is constructed from the equation system

$$\begin{aligned} LL^*\lambda &= d \\ u &= L^*\lambda \end{aligned} \tag{4.2}$$

The main point is that $LL^* \in \mathbb{R}^{n\times n}$ and it is thus easy to check whether $d \in \operatorname{Im}(LL^*)$ and in that case to find a suitable λ. In particular if $\operatorname{Ker} LL^* = 0$ then $u = L^*(LL^*)^{-1}d$.

In order to see that (4.2) in fact gives the minimum norm solution of (4.1) we take a solution $u = L^*\lambda + u_{null}$, where $u_{null} \in \operatorname{Ker}(L)$. Then

$$\begin{aligned} \|u\|_{L_2}^2 &= \|L^*\lambda\|_{L_2}^2 + 2\,\langle L^*\lambda, u_{null}\rangle_{L_2} + \|u_{null}\|_{L_2}^2 \\ &= \|L^*\lambda\|_{L_2}^2 + 2\,\langle \lambda, Lu_{null}\rangle_{L_2} + \|u_{null}\|_{L_2}^2 \\ &= \|L^*\lambda\|_{L_2}^2 + \|u_{null}\|_{L_2}^2 \geq \|L^*\lambda\|_{L_2}^2 \end{aligned}$$

with equality if $u_{null} = 0$.