KTH Matematik

Solution to Exam March 18, 2011 in SF2832 Mathematical Systems Theory.

For reference only

Allowed material: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory, Per Enqvist, Exercises in Mathematical Systems Theory, your own class notes, and β mathematics handbook.

Solution methods: All conclusions should be carefully motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 45 points credit (including your still valid bonus) to pass the exam. The other grade limits are listed on the course home page.

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1. Determine if each of the following statements is true or false. You must justify your answers. All matrices involved are assumed to be constant matrices unless otherwise specified.

(*a*) Consider $\dot{x} = Ax,\ y = Cx$. If $x_0 \notin ker\ \Omega$, then $e^{At}x_0 \notin ker\ \Omega,\ \forall t \geq 0.$ ...(5p)
True. Otherwise, $e^{AT}x_0 \in ker\ \Omega$ would implies $e^{-AT}e^{AT}x_0 \in ker\ \Omega$ by Taylor expansion.

(*b*) Consider a n-dimensional time-invariant controllable system $\dot{x} = Ax + Bu$. Given $x_0,\ x_1$ and $T > 0$, there are infinitely many continuous control $u(t)$ that transfer the state of the system from $x(0) = x_0$ to $x(T) = x_1$. .... (5p)
True. For example, we can let $u = Fx + v_F(t)$, where $v_F(t)$ is the minimum energy control for the feedback system.

(*c*) Suppose A is a stable matrix. For any positive definite matrix P, $-(A^TP+PA)$ is at least positive semi-definite. .... (5p)
False, according to a remark in the lecture notes (one can easily see this by taking $P = I,\ A = \begin{pmatrix} 0 & 2 \\ -1 & -1 \end{pmatrix}$).

(*d*) Suppose (A, b, c) is a minimal realization of a SISO transfer function $r(s) = \frac{n(s)}{d(s)}$, then $(c, A+bF)$ will still be observable for any F if and only if $n(s)$ is constant. (5p)
True, since there is no zero/pole cancellation in this case. Thus the closed-loop realization will remain minimum.

2. Consider a SISO system:

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx, \end{aligned}$$

where

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & -A_1 \end{bmatrix}, \; b = \begin{bmatrix} b_1 \\ b_1 \end{bmatrix}, \; c = \begin{bmatrix} c_1 & 0 & c_3 & 0 \end{bmatrix},$$

and the 2×2 matrix A_1 is such that $A_1^2 = 0$.

(a) Find the state transition matrix e^{At}, namely, express e^{At} as a matrix polynomial in terms of $A_1,\ A_2$. .... (6p)
Since $A^2 = \begin{pmatrix} 0 & A_1A_2 - A_2A_1 \\ 0 & 0 \end{pmatrix}$, $A^3 = \begin{pmatrix} 0 & -A_1A_2A_1 \\ 0 & 0 \end{pmatrix}$, $A^4 = 0$,

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3.$$

(b) Suppose (A_1, b_1) is controllable and is already in controllable (reachable) canonical form, and $A_2 = [b_1, A_1b_1]$. Show (A, b) is controllable. .... (8p)
Then $A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $b_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The conclusion follows.

(c) Under the same assumptions as in (b), find conditions on $c_1,\ c_3$ such that under any pole placement, $(c, A + bF)$ is still observable. .... (6p)
This is to find c such that $c(sI - A)^{-1}b_1$ has a constant numerator, which gives $c = (k\ 0\ 2k\ 0),\ k \neq 0$.

3. Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{\beta}{(s+1)(s+2)} \\ \frac{1}{s+1} & \frac{\gamma}{(s+1)(s+2)} \end{bmatrix},$$

where $\beta,\ \gamma$ are constants.

(a) Find the standard reachable realization. .... (10p)
We should consider two different cases: 1. β and γ are not both zero; 2. they are both zero. In case 1, $\chi(s) = s^2 + 3s + 2$, while in case 2, $\chi(s) = s + 1$. The rest is omitted.

(b) Can the realization in (a) also be observable and why? .... (5p)
No. In case 1, the realization has dimension 4, while the McMillan degree is 2 if $\beta = \gamma$, and 3 otherwise. In case 2, the realization has dimension 2 , while the McMillan degree is 1.

(c) For the case $\beta = \gamma = 0$, find a minimal realization of $R(s)$. .... (5p)
From

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \; B = I_2, \; C = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

we can derive a minimum realization

$$A = -1, \; B = 1 \; (or \; [1\ 0]), \; C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

4. In Chapter 3 we derived a minimum energy control for transferring the system from one state to another state. It is intuitive that the shorter time it takes to reach a given state the more energy the control spends. In this problem we show mathematically this is true.

Consider a controllable system

$$\dot{x} = Ax + Bu.$$

Suppose we want to transfer an arbitrary x_0 at $t = 0$ to the origin at $t = t_1$. It is proven that

$$\hat{u} = -B^T e^{A^T(t_1-t)} W^{-1}(0, t_1) e^{At_1} x_0$$

where W is the reachability Gramian, is a feasible control that further minimizes among all feasible controls

$$J(u) = \int_0^{t_1} u^T(s)u(s)ds.$$

We denote $J(\hat{u}) = x_0^T L(t_1) x_0$.

(*a*) Show $W(0, t)$ satisfies $AW + WA^T + BB^T = e^{At}BB^Te^{A^Tt}$. . . . . (4p)
Use the same technique as on p. 33 of the lecture notes, where we let $Q = BB^T$ and replace A by A^T.

(*b*) Show that for any $x_0 \neq 0$, $x_0^T L(t_2)x_0 < x_0^T L(t_1)x_0$, $t_2 > t_1$. (Hint: for a nonsingular matrix $M(t)$, $\frac{d}{dt}(M^{-1}(t)) = -M^{-1}\dot{M}M^{-1}$.) . . . . (8p)
We can derive $L(t) = exp(A^T t)W^{-1}(0, t)exp(At)$.
Since $x_0^T \dot{L}(t)x_0 = -x_0^T L(t)exp(-At)BB^T exp(-A^T t)L(t)x_0$, integrating both sides from t_1 to t_2 gives the conclusion.

(*c*) It is obvious that $\lim_{t_1\to\infty} L(t_1) = 0$ if A is a stable matrix. What is $\lim_{t_1\to\infty} L(t_1)$ if $-A$ is a stable matrix? (Hint: L would be determined if L^{-1} is.) . . . . (8p)
From the equality in (a), we have $-AL^{-1} - L^{-1}A^T - e^{-At}BB^Te^{-A^Tt} = -BB^T$.
Then we can easily see what will happen as $t \to \infty$ if $-A$ is a stable matrix.

5. (*a*) Let x be the outcome of a random variable with distribution $N(0, \alpha^2)$ (i.e., $E\{x\} = 0$, $E\{x^2\} = \alpha^2$). We would like to determine the value of x by the following set of noisy measurements

$$y(t) = x + tw(t) \text{ for } t = 0, 1, \ldots, n, \ldots$$

where $w(t) \in N(0, \sigma^2)$ are independent of each other and of x. Let $\hat{x}_t = E^{H_{t-1}(y)}x(t)$, and $P(t) = E\{(x - \hat{x}_t)^2\}$. Show $P(t + 1) < P(t)$ and discuss what happens to $P(t)$ as $t \to \infty$. . . . . (10p)
Here we only write down the update equation for the covariance: $P^{-1}(t+1) = P^{-1}(t) + (\sigma^2t^2)^{-1}$, $P(0) = \alpha^2$. Thus $P(t)$ is strictly decreasing and the limit is greater than 0 since $\sum^\infty(\sigma^2t^2)^{-1}$ is finite.

(*b*) Consider $\dot{x} = Ax, y = Cx$. Show that the system is observable if and only if the only solution that satisfies $Cx(t) \equiv 0$ ($\forall t \geq 0$) is $x(t) \equiv 0$. . . . . (10p)
$Cx(t) \equiv 0$ implies $Cx^{(k)}(t) \equiv 0$, $k = 0, 1, \cdots$, i.e. $CA^kx(t) \equiv 0$, which is equivalent to $\Omega x(t) \equiv 0$. Thus, $x(t) \equiv 0$ is the only solution iff rank $\Omega = n$.

Good luck!