KTH Matematik

Tentamen i 5B1742 Mathematical Systems Theory
May 23 2006 8.00–13.00
Answers and solution sketches

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1. (a) The linerized dynamics is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

(b) The linearized system is a 2-dimensional Jordan block with eigenvalue at 0. It is therefore not stable.

(c) We get

$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(t)$$

(d) The discrete time system is also unstable because F is a 2-dimensional Jordan block with eigenvale at 1. This is not a surprise because sampling of a divergent system gives a divergent time series.

(e) We use the procedure in the lecture notes. We have

$$\Gamma = \begin{bmatrix} 0.5 & 1.5 \\ 1 & 1 \end{bmatrix}$$

and $t\Gamma = \begin{bmatrix} 0 & 1 \end{bmatrix}$ is solved by $t = \begin{bmatrix} 1 & -0.5 \end{bmatrix}$, which implies that $T = \begin{bmatrix} 1 & -0.5 \\ 1 & 0.5 \end{bmatrix}$. The transformed system has realization

$$\hat{F} = TFT^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \quad h = Tb = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The state feedback gain $g = \begin{bmatrix} 1 & -2 \end{bmatrix}$ gives the desired characteristic polynomial $\varphi(z) = z^2$ for the transformed system. In the original coordinates we use $k = gT = \begin{bmatrix} -1 & -1.5 \end{bmatrix}$.

(f) The state observer has the form

$$\hat{x}(t+1) = F\hat{x}(t) + Gu(t) + L(y(t) - C\hat{x}(t))$$

where $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Design of the observer gain such that the error dynamics $e(t+1) = (F - LC)e(t)$ has characteristic polynomial $\varphi_{F-LC}(z) = (z-0.5)^2$ can using the same procedure as in (e). We get $L = \begin{bmatrix} 1 & 0.25 \end{bmatrix}^T$.

2. (a) We have the factorization

$$\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \end{bmatrix} = \frac{1}{s^2+3s+2}\left(\begin{bmatrix} 2 & 1 \end{bmatrix} + s\begin{bmatrix} 1 & 1 \end{bmatrix}\right)$$

which gives the following matrices for the standard reachable realization

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & -3 & 0 \\ 0 & -2 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 1 & 1 & 1 \end{bmatrix}$$

(b) For problem (b) and (c) we can either compute the McMillan degree as $\delta(R) = \text{rank}(H_r)$ and then use Ho's algorithm or we can show that the standard reachable realization is not observable and then use Kalman decomposition to remove the unobservable states.

Here we take the second approach. The observability matrix is

$$\Omega = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ -2 & -2 & -1 & -2 \\ 2 & 4 & 1 & 4 \\ -2 & -8 & -1 & -8 \end{bmatrix}$$

By using elementary column operations we get the factorization

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ -2 & -2 & -1 & -2 \\ 2 & 4 & 1 & 4 \\ -2 & -8 & -1 & -8 \end{bmatrix} \begin{bmatrix} 1 & 0 & -0.5 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ -2 & -2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ -2 & -8 & 0 & 0 \end{bmatrix}$$

From this it follows that

$$\text{Ker}\Omega = \text{span}\left\{ \begin{bmatrix} -0.5 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

so two states can be removed using Kalman decompostion.

(c) $\mathbf{R}^4 = \text{Ker}(\Omega) \oplus V_o$, where

$$V_o = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

By making the change of coordinates

$$x = \underbrace{\begin{bmatrix} -0.5 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{T} \begin{bmatrix} z_{\bar{o}} \\ z_o \end{bmatrix}$$

we get the realization

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & -3 & 0 \\ 0 & -2 & 0 & -3 \end{bmatrix} \begin{bmatrix} -0.5 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & -2 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$\tilde{B} = T^{-1}B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\tilde{C} = CT = \begin{bmatrix} 0 & 0 & 2 & 1 \end{bmatrix}$$

From this we immediately get the following minimal realization by removing the unobservable states

$$\dot{z}_o = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} z_o + \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} z_o$$

3. (*a*) Let the initial state be $x_0 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. Then the solution to the system equation on the time interval $[0, T/2)$ is

$$x(t) = \begin{bmatrix} t + \int_0^t u(\tau)d\tau \\ 1 \end{bmatrix}, \qquad t \in [0, T/2),$$

which can never become 0.

(*b*) The reachability Gramian is $t \in (T/2, T)$

$$W(0,t) = \int_0^{0.5T} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} d\tau + \int_{0.5T}^{t} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} d\tau = \begin{bmatrix} 0.5T & 0 \\ 0 & t - 0.5T \end{bmatrix}$$

which is invertible when $t > T/2$. Hence, any initial condition can be steered to zero according to Theorem 3.1.4 in Lindquist and Sand.

4. (*a*) The Riccati equation becomes

$$\dot{p} = e^{-2t}p^2, \quad p(1) = 1.$$

By separation of variables we get (c is an arbitrary constant)

$$\frac{dp}{p^2} = e^{-2t}dt \quad \Leftrightarrow \quad -\frac{1}{p} = -\frac{1}{2}e^{-2t} - c \quad \Leftrightarrow \quad p(t) = \frac{1}{c + 0.5e^{-2t}}$$

The boundary condition $p(1) = 1$ is satisfied if $c = 1-0.5e^{-2}$, hence the optimal control is

$$u(t) = -e^{-t}p(t)x(t) = -\frac{e^{-t}}{1+0.5(e^{-2t}-e^{-2})}x(t)$$

(*b*) This problem can be solved using the reachability theory. The optimal control problem can be formulated as

$$\min \|u\|^2 \quad \text{subj. to} \quad Lu = d \quad \text{where} \quad \begin{cases} d = -1 \\ Lu = \int_0^1 e^{-\tau}u(\tau)d\tau \end{cases}$$

The solution is (it is easy to show that $(L^*x)(t) = e^{-t}x$ for any real number x)

$$u(t) = L^*(LL^*)^{-1}d = -\frac{2e^{-t}}{1-e^{-2}}, \quad t \in [0,1]$$

5. (*a*) As in Lindquist and Sand, we embed the problem in the Kalman filter set-up using the dynamical system

$$\begin{aligned} x(t+1) &= x(t), \quad Ex(t) = 0, \quad Ex(0)^2 = p_0 \\ y(t) &= Cx(t) + w(t), \quad C = \mathbf{1} \end{aligned}$$

where p_0 models our uncertainty in the knowledge of d. The best linear least squares estimator is given by the Kalman filter

$$\hat{x}(t+1) = \hat{x}(t) + k(t)(y(t) - C\hat{x}(t)), \quad \hat{x}(0) = 0$$

where

$$\begin{aligned} k(t) &= p(t)\mathbf{1}^T(p(t)\mathbf{1}\mathbf{1}^T + I)^{-1} \\ p(t+1) &= p(t) - p(t)^2\mathbf{1}^T(p(t)\mathbf{1}\mathbf{1}^T + I)^{-1}\mathbf{1}, \quad p(0) = p_0 \end{aligned}$$

(*b*) By using the Sherman-Morrison formula in the hint we get

$$p(t+1) = (p(t)^{-1} + \mathbf{1}^T\mathbf{1})^{-1} = (p(t)^{-1} + N)^{-1}$$

which implies

$$p(t+1) = \frac{p(t)}{1+Np(t)} = \frac{p_0}{1+(t+1)Np_0}$$

where the second expression can be proven using induction. This shows that the variance decreases faster the more sensors we have available.

Similarly the Kalman gain can be simplified to

$$k(t) = \frac{p(t)}{1+Np(t)}\mathbf{1}^T = \frac{p_0}{1+(t+1)Np_0}\mathbf{1}^T$$

(c) At each node we can model the sensor information as the following system

$$\begin{aligned} x_i(t+1) &= x_i(t), \quad Ex_i(t) = 0, \quad Ex_i(0)^2 = p_0 \\ z_i(t) &= C_i x_i(t) + D_i w_i(t), \end{aligned}$$

where $w_i(t)$ is a noise with $Ew_i = 0$, $Ew_i(t)w_i(t)^T = I_i \delta_{t,s}$, where I_i is an identity matrix of size n_i, where n_i is the number of elements in N_i. Moreover,

$$\begin{aligned} C_i &= \sum_{j \in N_i} c_{i,j} \\ D_i &= \begin{bmatrix} c_{i,j_1} & \dots & c_{i,j_{n_i}} \end{bmatrix} \end{aligned}$$

where $j_1, \dots, j_{n_i}$ are the indices in N_i. The Kalman filter becomes (using the same calculations as in (a) and (b))

$$\hat{x}_i(t+1) = \hat{x}_i(t) + k_i(z_i(t) - C_i \hat{x}_i(t))$$

where

$$\begin{aligned} k_i(t) &= p_i(t) C_i (p_i C_i^2 + D_i D_i^T)^{-1} \\ p_i(t+1) &= p_i(t) - p_i(t)^2 C_i (C_i^2 p_i(t) + D_i D_i^T)^{-1} \end{aligned}$$