SYSTEMTEORI - ÖVNING 3

1. Stability of linear systems

Exercise 3.1 (LTI system). Consider the following matrix:

$$A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$$

Use the Lyapunov method to determine if A is a stability matrix:

a): in continuous time sense
b): in discrete time sense

Remark 1.1. The eigenvalues of A are computed as

$$\det\,(\lambda I - A) = 0 \Rightarrow \lambda_{1,2} = 1, -2.$$

Remark 1.2. In Matlab, `lyap` and `dlyap` are the solvers for continuous-time and discrete-time Lyapunov equations, respectively.

a): The Lyapunov equation (LE) is given by:

$$A^T P + PA + Q = 0$$

Take $Q = I$ which is a positive definite matrix as required by Cor. 4.1.12 in the compendium. Solve LE for P defined as [1] :

$$P = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Then, we have:

$$\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix} + \begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

$$\Rightarrow \left\{ \begin{array}{rcl} 4b+1 & = & 0 \\ a-b+2c & = & 0 \\ 2b-2c+1 & = & 0 \end{array} \right. \Rightarrow P = \frac{1}{4}\begin{pmatrix} -3 & -1 \\ -1 & 1 \end{pmatrix} \not> 0$$

Unfortunately, the matrix P is not positive definite, and so the matrix A is not a stability matrix in the continuous time. This result is consistent with Remark 1.1, since there is an eigenvalue at 1 (which has a positive real part).

b): Consider the discrete Lyapunov equation (DLE):

$$A^T PA - P + Q = 0.$$

Take $Q = I$ and solve DLE for P defined as:

$$P = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

[1] Note that P must be symmetric.

Then, we have:

$$\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} - \begin{pmatrix} a & b \\ b & c \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0.$$

$$\Rightarrow \begin{cases} -a+4c+1 & = & 0 \\ b-2c & = & 0 \\ a-2b+1 & = & 0 \end{cases}$$

In this case the system of equation is not well defined and there are infinite solutions P. A general solution is given by:

$$P = \begin{pmatrix} \varepsilon & (\varepsilon+1)/2 \\ (\varepsilon+1)/2 & (\varepsilon-1)/4 \end{pmatrix}$$

where $\varepsilon > 0$. By using Sylvester test, we derive that $P \not> 0$ and so the matrix A is not a stability matrix in discrete time sense.

Remark 1.3. In this example, the DLE does not have a unique solution, because A has an eigenvalue on the unit circle.

Example. Build an example of a system with poles on the imaginary axis, and show that the multiplicity of the poles influences the stability.

Let us consider the two easiest possibilities:

1): two poles in $z = 0$ with two Jordan blocks of size 1-by-1:

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

2): two poles in $z = 0$ with one Jordan block of size 2-by-2:

$$A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We have that:

$$e^{A_1 t} = I \qquad \Rightarrow \quad x_1(t) = x_0$$

$$e^{A_2 t} = I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t \quad \Rightarrow \quad x_2(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} x_0 \rightarrow \text{unbounded as } t \rightarrow \infty.$$

Exercise from tenta 031022.

(a): Consider the following system:

$$\dot{x}(t) = Ax(t)$$

Prove the following theorem:

Theorem: All the eigenvalues of the matrix A have real part smaller than $-c$, where $c > 0$, iff for every symmetric and positive definite matrix Q, there exists a unique symmetric, positive definite matrix P such that:

$$A^T P + PA + 2cP = -Q.$$

Let us consider the following matrix:

$$F := A + cI.$$

It is easy to see that eig(F)=eig(A)+c. The Lyapunov theorem can be applied to F. That is, F is a stability matrix iff for every symmetric positive definite matrix Q, there exists a unique symmetric positive definite matrix P such that LE is satisfied:

$$F^T P + FP = -Q.$$

But, if we use the fact that $F = A + cI$ we get:

$$F^T P + FP = A^T P + PA + 2cP = -Q.$$

(b): The matrices A_1, A_2, and A_3 have the following characteristic polynomials:

$$\begin{aligned}
\chi_1(\lambda) &= \det(\lambda I - A_1) = \lambda^4 + 2c^2\lambda^2 + c^4 \\
\chi_2(\lambda) &= \det(\lambda I - A_2) = \lambda^4 + 2c\lambda^3 + 2c^2\lambda^2 + 2c^3\lambda + c^4 \\
\chi_3(\lambda) &= \det(\lambda I - A_3) = \lambda^4 - c^4
\end{aligned}$$

where c is a real positive parameter. What can we say about the stability of the following systems for different values of the parameter c:

$$\dot{x}(t) = Ax(t) \quad \text{and} \quad x(n+1) = Ax(n)$$

respectively?

Let us rewrite the polynomials in a smarter form:

$$\begin{array}{lll}
\chi_1(\lambda) = (\lambda^2 + c^2)^2 = 0 & \Leftrightarrow & \lambda = \pm ic \quad \text{double roots} \\
\chi_2(\lambda) = (\lambda^2 + c^2)(\lambda + c)^2 = 0 & \Leftrightarrow & \lambda = \pm ic \text{ and } \lambda = -c \quad \text{double root} \\
\chi_3(\lambda) = (\lambda^2 + c^2)(\lambda^2 - c^2) = 0 & \Leftrightarrow & \lambda = \pm ic \text{ and } \lambda = \pm c
\end{array}$$

Now, consider the two different situations:

i) $\dot{x} = A_k x$:

$k = 1$**:** Stable but not asymptotically stable, **or** unstable for all $c > 0$. For example, the following two matrices have the same characteristic polynomial $\chi_1(\lambda)$:

$$B = \begin{bmatrix} ic & 0 & 0 & 0 \\ 0 & ic & 0 & 0 \\ 0 & 0 & -ic & 0 \\ 0 & 0 & 0 & -ic \end{bmatrix}, \quad C = \begin{bmatrix} ic & 1 & 0 & 0 \\ 0 & ic & 0 & 0 \\ 0 & 0 & -ic & 1 \\ 0 & 0 & 0 & -ic \end{bmatrix}.$$

Note that B is stable and C is unstable. In this way, when the characteristic polynomial have roots on the boundary of the stability region, we need to see the Jordan form of A to judge the stability of the system.

$k = 2$**:** Stable but not asymptotically stable for all $c > 0$.

$k = 3$**:** Unstable for all c.

ii) $x(n+1) = A_k x(n)$:

In this case the systems are unstable for $c > 1$, and asymptotically stable if $c < 1$. If $c = 1$ we have that:

$k = 1$**:** Stable but not asymptotically stable, **or** unstable.

$k = 2$**:** Stable but not asymptotically stable, **or** unstable.

$k = 3$**:** Stable, but not asymptotically stable.

Example (LTV system). Consider the following LTV system:

$$\dot{x}(t) = \underbrace{\begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix}}_{A(t)} x(t).$$

Show that it is not stable even if all the eigenvalues of the matrix $A(t)$ have negative real parts for any t.

It is easy to see that, for any t, the matrix $A(t)$ has eigenvalues at -1 (with multiplicity two) , that is, eigenvalues with negative real parts.

In order to prove instability of this system, let us compute the solution of the system, assuming

$$x(t_0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}.$$

From the second equation, we obtain

$$x_2(t) = e^{-(t-t_0)}x_{20}.$$

Substituting this into the first equation, we have

$$\begin{aligned} &\dot{x}_1(t) = -x_1(t) + e^{2t} \cdot e^{-(t-t_0)}x_{20}, \\ &\Rightarrow \dot{x}_1(t) = -x_1(t) + e^{(t+t_0)}x_{20}, \\ &\Rightarrow \begin{cases} x_1(t) &= e^{-(t-t_0)}x_{10} + \int_{t_0}^{t} e^{-(t-s)} \cdot e^{(s+t_0)}x_{20}ds \\ &= e^{-(t-t_0)}x_{10} + e^{(t_0-t)}\dfrac{1}{2}(e^{2t} - e^{2t_0})x_{20} \\ &= e^{-(t-t_0)}x_{10} + \dfrac{1}{2}(e^{t+t_0} - e^{3t_0-t})x_{20}. \end{cases} \end{aligned}$$

Therefore, $x_1(t)$ is unbounded when t goes to infinity.

2. Kalman decomposition

Example (Tenta 010829). Consider the following system:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 2 & 4 & 3 \\ 4 & 2 & 3 \\ 4 & 4 & 4 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & -2 \end{bmatrix} u \\ y &= \begin{bmatrix} 2 & 2 & -3 \end{bmatrix} x \end{aligned} \tag{2.1}$$

(a): Determine the basis of the subspaces for the Kalman decomposition.
(b): Transform the system in the corresponding form.

(a): In the Kalman decomposition, we consider 4 subspaces[2]:

(1) The reachable and unobservable subspace

$$V_{\bar{o}r} := \mathrm{Im}\Gamma \cap \ker\Omega. \tag{2.2}$$

(2) Any complementary subspace V_{or} to $V_{\bar{o}r}$ in $\mathrm{Im}\Gamma$, i.e.,

$$\mathrm{Im}\Gamma = V_{\bar{o}r} \oplus V_{or}. \tag{2.3}$$

This space will be essential to the input-output relation of the system.
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[2] In the lecture note, in page 45, Theorem 5.2.1 is stated with the notation “+”, but it should be corrected to the direct sum “$\oplus$.”

(3) Any complementary subspace $V_{\bar{o}\bar{r}}$ to $V_{\bar{o}r}$ in $\ker\Omega$, i.e.,

$$\ker\Omega = V_{\bar{o}r} \oplus V_{\bar{o}\bar{r}}. \tag{2.4}$$

(4) Any complementary subspace $V_{o\bar{r}}$ to $\mathrm{Im}\Gamma + \ker\Omega$ in $\mathbb{R}^n$, i.e.,

$$\mathbb{R}^n = V_{\bar{o}r} \oplus V_{or} \oplus V_{\bar{o}\bar{r}} \oplus V_{o\bar{r}}. \tag{2.5}$$

Remark 2.1. Note that we are using neither the terminology "unreachable subspace" nor "observable subspace." All the arguments are done with only "reachable subspace" and "unobservable subspace." The reason for avoiding the terminologies "unreachable subspace" or "observable subspace" is that these subspaces are not uniquely determined. However, for example, the states $x_{\bar{r}o}$ are indeed not reachable but observable. We use the subscripts "$\bar{r}$" and "o" to mean that the subspaces are in a complement of the reachable subspace, and in a complement of the unobservable subspace, respectively.

First, let us determine the reachability matrix Γ:

$$\Gamma = [B, AB, A^2B] = \begin{bmatrix} 1 & 1 & -2 & 0 & 4 & 0 \\ -1 & 1 & 2 & 0 & -4 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, we have that the reachable space $\mathcal{R}$ is given by:

$$\mathcal{R} = \mathrm{Im}\ \Gamma = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}.$$

To determine the unobservable subspace, we first compute the observability matrix Ω:

$$\Omega = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, the unobservable subspace, $\ker\ \Omega$, is given by:

$$\ker\Omega = \{x : \ 2x_1 + 2x_2 - 3x_3 = 0\} = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} \right\}.$$

Finally, we can determine the previous 4 subspaces by combining the two subspaces $\mathcal{R}$ and $\ker\ \Omega$. To obtain $V_{\bar{o}r}$, we need to find the intersection of two subspaces $\mathrm{Im}\Gamma = \{v_1, \cdots, v_l\}$ and $\ker\Omega = \{u_1, \cdots, u_m\}$. Any vector on the intersection must be expressed as a linear combination of v_k, as well as a linear combination of u_k:

$$\sum_{k=1}^{l} \alpha_k v_k = \sum_{k=1}^{m} \beta_k u_k, \quad \alpha_k, \beta_k \in \mathbb{R}.$$

Finding such α_k and β_k is equivalent to finding the kernel space of

$$[v_1, \cdots, v_l, u_1, \cdots, u_m].$$

Try to find the intersection subspace in this example by yourselves!

In this example,

$$
\begin{aligned}
V_{\bar{o}r} &= \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right\}, \\
V_{or} &= \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}\right\}, \\
V_{\bar{o}\bar{r}} &= \operatorname{span}\left\{\begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}\right\}, \\
V_{o\bar{r}} &= \{0\}.
\end{aligned}
$$

(b): From the bases of the four subspaces in (a), we form a nonsingular matrix:

$$
T := \left[\begin{array}{c|c|c} 1 & 1 & 0 \\ -1 & 1 & 3 \\ 0 & -2 & 2 \end{array}\right].
$$

By a variable change $x = Tz$, we obtain

$$
\begin{aligned}
\dot{z} &= \underbrace{\left[\begin{array}{c|c|c} -2 & 0 & 18 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 10 \end{array}\right]}_{\tilde{A}=T^{-1}AT} z + \underbrace{\left[\begin{array}{cc} 1 & 0 \\ \hline 0 & 1 \\ \hline 0 & 0 \end{array}\right]}_{\tilde{B}=T^{-1}B} u \\
y &= \underbrace{\left[\begin{array}{c|c|c} 0 & 10 & 0 \end{array}\right]}_{\tilde{C}=CT} z.
\end{aligned}
$$

Note that the transformed A, B and C matrices are in the following form:

$$
\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} \\ 0 & \tilde{A}_{22} & 0 & \tilde{A}_{24} \\ 0 & 0 & \tilde{A}_{33} & \tilde{A}_{34} \\ 0 & 0 & 0 & \tilde{A}_{44} \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ 0 \\ 0 \end{bmatrix}, \ \tilde{C} = \begin{bmatrix} 0 & \tilde{C}_2 & 0 & \tilde{C}_4 \end{bmatrix}.
$$

Note also that, in this example, the fourth row/column blocks in $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ do not appear, since $V_{o\bar{r}} = \{0\}$.

The system:

$$
\begin{aligned}
\dot{\tilde{x}} &= \tilde{A}_{22}\tilde{x} + \tilde{B}_2 u \\
y &= \tilde{C}_2 \tilde{x}
\end{aligned} \tag{2.6}
$$

is completely reachable and completely observable, and it has the same transfer function as the original system (2.1). The realization given in (2.6) is called a *minimal realization.*

For this particular example, we have

$$
\begin{aligned}
G(s) &= C(sI - A)^{-1}B = \begin{bmatrix} 0 & 10/s \end{bmatrix}, \\
\tilde{G}(s) &= \tilde{C}_2(sI - \tilde{A}_{22})^{-1}\tilde{B}_2 = \begin{bmatrix} 0 & 10/s \end{bmatrix}.
\end{aligned}
$$