KTH Matematik

Solution to Homework 3
Mathematical Systems Theory, SF2832
Fall 2018
You may use min(5,(your score)/4) as bonus credit on the exam.

1. Consider a state space realization (A, b, c) as follows

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} c_1 & c_2 & 1 \end{bmatrix} x,$$

where $c_1,\ c_2$ are constants.

(a) For what $c_1,\ c_2$ is the system observable? .... (1p)
Solution: The poles are roots of $s^3 + s^2$ and zeros are roots of $s^2 + c_2 s + c_1$. Observability implies that there is no zero-pole cancellation. The rest is omitted.

(b) Design a feedback controller $u = kx$ such that dimension of the unobservable subspace for $(c, A + bk)$ is maximized while $A + bk$ is guaranteed to have one eigenvalue at -1. .... (1p)
Solution: The dimension of the unobservable subspace is maximized when all zeros are canceled by the poles. Thus the desired closed-loop poles should be roots of $(s + 1)(s^2 + c_2 s + c_1)$. The rest is omitted.

(c) Design a feedback control $u = kx$ that makes $y(t) = 0,\ \forall t \geq 0$ if initially $y(0) = 0$. For the closed-loop system, discuss conditions on c_1, c_2 such that $\lim_{t\to\infty} x(t) = 0$ when $y(t) = 0,\ \forall t \geq 0$. .... (3p)
Solution: $y = x_3 + c_2 x_2 + c_1 x_1 \equiv 0$ implies that $\dot{y} = -x_3 + u + c_2 x_3 + c_1 x_2 \equiv 0$. Then $u = x_3 - c_2 x_3 - c_1 x_2$. $\lim_{t\to\infty} x(t) = 0$ if both c_1, c_2 are positive.

2. Consider a state space system as follows

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \alpha x_1 + u$$
$$y = x_2,$$

where α is a constant. Let $P(t, t_1)$ where $0 \leq t \leq t_1$ denote the solution to the dynamical Riccati equation associated with the following optimal control problem

$$min\ J = \int_0^{t_1} (y^2 + u^2) dt.$$

(a) For what α is $P(t, t_1)$ positive definite $\forall t < t_1$? .... (2p)
Solution: $\alpha \neq 0$.

(*b*) For $\alpha = 0$ compute $\lim_{t_1 \to \infty} P(0, t_1)$ and give the reason why the limit exists but is not positive definite. ....(3p)

Solution: When $\alpha = 0$, The cost function does not depend on x_1. Thus $p_{11}(t) = p_{12}(t) = p_{21}(t) \equiv 0$ for any $t_1 > 0$. $p_{22}(t)$ is solved by minimizing the cost function subject to $\dot{x}_2 = u$. The rest is omitted.

3. Consider the Dynamical Riccati equation

$$\dot{P} = -A^T P - PA + PBB^T P - Q$$
$$P|_{t=t_1} = S,$$

where $Q \geq 0,\ S \geq 0$ and we use $P(t, t_1, S)$ to denote the unique solution (which implies that $P(t_1, t_1, S) = S$).

(*a*) Show that $P(t_0, t_1, 0) \leq P(t_0, \bar{t}_1, 0)$ for any $t_0 \leq t_1 \leq \bar{t}_1$ (we say $P_1 \leq P_2$ if $P_2 - P_1 \geq 0$). ....(3p)

Solution: Note that $x_0^T P(t_0, t_1, S) x_0$ is the optimal cost over the time interval $[t_0, t_1]$. The cost will be higher if the time interval is longer, thus $P(t_0, t_1, 0) \leq P(t_0, \bar{t}_1, 0)$ for any $t_0 \leq t_1 \leq \bar{t}_1$.

(*b*) Assume that (A, B) is controllable. Show that $P_a = P(t, t_1, P_a)$ for any $t \leq t_1$, where $P_a = \lim_{t_1 \to \infty} P(0, t_1, 0)$. ....(2p)

Solution: Clearly P_a is constant and is the solution to the corresponding ARE. When $S = P_a$, P_a satisfies both the terminal condition and the dynamical Riccati equation. Thus $P_a = P(t, t_1, P_a)$.

4. Consider a one-dimensional system

$$x(t+1) = ax(t) + v(t)$$
$$y(t) = x(t) + w(t),$$

where $a \neq 0,\ v, w$ are uncorrelated white noises, with covariances $\sigma,\ r$ respectively.

Solution: This problem has appeared before as a homework problem. The solution is thus omitted.

(*a*) Design a Kalman filter $\hat{x}(t)$ for $x(t)$. ....(1p)

(*b*) Express the covariance matrix $p(t) = E\{(x(t) - \hat{x}(t))^2\}$ in terms of $a, \sigma,\ r$.(2p)

(*c*) What is $a - ak(t)$ as $t \to \infty$ (where $k(t)$ is the Kalman gain)? ....(2p)