KTH Matematik

Solution to Homework 2
Mathematical Systems Theory, SF2832
Fall 2018
You may use min(5,(your score)/4) as bonus credit on the exam.

1. Consider the pair (c, A), where

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$
$$c = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

For what a_1 and a_2 the Lyapunov equation $A^T P + PA + c^T c = 0$ (where P is assumed to be symmetric)

(a) has a positive definite solution? .... (1p)
Solution: $a_1 < 0, a_2 < 0$ and $a_1 \neq a_2$.

(b) has a negative definite solution ($-P$ is positive definite)?
Solution: Omitted. .... (1p)

(c) has no solution ? .... (2p)
Solution: $a_1 + a_2 = 0$.

2. Given the following system

$$\dot{x} = Ax + bu = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$
$$y = cx = \begin{bmatrix} c_1 & c_2 \cdots & c_{n-1} & 1 \end{bmatrix} x.$$

(a) Show that if $ce^{At}x_0 = 0, \ \forall t \geq 0$ implies that x_0 must be 0, then (c, A) is observable. For the case $c_i = 0, \ i = 1, \cdots, n-1$, discuss condition on $a_i, \ i = 1, \cdots, n$ such that the system is observable. .... (2p)
Solution: $ce^{At}x_0 \equiv 0$ implies $(ce^{At}x_0)^{(k)} \equiv 0$, i.e. $cA^k e^{At}x_0 \equiv 0, \ k = 0, 1, \ldots$. Then $\Omega e^{At}x_0 \equiv 0$ and the conclusion follows. For the given case, $a_1 \neq 0$ is the condition for observability.

(b) For the case $c_i = 0, \ i = 1, \cdots, n-2$ and $c_{n-1} = \gamma$, find a feedback control $u = kx$ such that the dimension of the unobservable subspace for the closed-loop system is maximized, where the closed-loop system is $\dot{x} = (A + bk)x, \ y = cx$. (2p)
Solution: The desired poles should be defined by $\phi(s) = (s^{n-1} + \gamma s^{n-2})(s + d)$ where d is arbitrary. Then the feedback matrix k follows.

(c) For what n and γ, there is a feedback control solving (b) that is also a stabilizing control? Namely all eigenvalues of $A + bk$ have negative real parts. ...... (2p)
Solution: $n = 2$ and $\gamma > 0$.

3. Consider

$$R(s) = \begin{bmatrix} \frac{k}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix},$$

where k is a constant.

(a) Determine the standard reachable realization of $R(s)$. .... (1p)
Solution: Omitted

(b) Determine the standard observable realization of $R(s)$. .... (2p)
Solution: Omitted

(c) What is the McMillan degree of $R(s)$? .... (2p)
Solution: $\delta(R) = 1$ if $k = 1$ otherwise $\delta(R) = 2$.

4. Suppose that the following is a realization of a given $R(s)$:

$$(A, B, C) = \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \right)$$

(a) Find a feedback control $u = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} x$ that assigns the closed-loop poles to $\{-1, -1, -1, -2\}$, and contains as few non-zero elements in k_1 as possible. .... (2p)
Solution: Since u_2 alone would make the system controllable, we can set $k_1 = 0$. Then an easy way to solve the pole placement problem is to convert the single input system into the standard controllable form, for which we obtain that $c = [1 \ -2 \ 3 \ 2]$ (do not confuse this c with the output matrix given for the realization). The rest is omitted.

(b) Is the realization minimal? If not, use Kalman decomposition to find a minimal realization. .... (3p)
Solution: No, since it is not observable. Here is one minimal realization: $\dot{x}_1 = x_2 + u_1 + u_2, \dot{x}_2 = -u_1 + u_2, y = x_1$.