KTH Matematik

Solution to Homework 1
Mathematical Systems Theory, SF2832
Fall 2018
You may use min(5,(your score)/4) as bonus credit on the exam.

1. (*a*) Solve $\dot{x}(t) = \begin{bmatrix} -1 & 1 & t \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t)$, $x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ . . . . (2p)

Solution: Obtain first $x_3(t) = e^t$ and plug in it to the first two equations. The rest is omitted.

(*b*) Solve $\dot{x}(t) = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x(t)$, $x(0) = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}$, where ω is a positive constant, and show that $x_1(t) = a\sin(\omega t + b)$ where a, b are functions of $x(0)$. . . . .(2p)

Solution: $a = \|x(0)\|$, $b = \arcsin \frac{x_1^0}{a}$.

2. Consider an $n \times n$ matrix

$$A(t) \quad = \quad e^{A_1 t} A_2 e^{-A_1 t},$$

where A_1, A_2 are constant matrices.

(*a*) Show that $\Psi(t) = e^{A_1 t} e^{A_2 t} e^{-A_1 t}$ is a fundamental matrix to $\dot{x} = A(t)x$ if $A_1 A_2 = A_2 A_1$. . . . . (2p)

Solution: We only need to show $\Psi(t)$ is nonsingular and $\dot{\Psi}(t) = A(t)\Psi(t)$.

(*b*) For the general case, find a fundamental matrix in the form $\Psi(t) = e^{A_1 t} e^{A_3 t}$ and specify your A_3 in terms of A_1, A_2. . . . .(2p)

Solution: $\Psi(t) = e^{A_1 t} e^{(A_2 - A_1)t}$

3. Consider

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x. \end{aligned}$$

Show that controllability and observability of linear time-varying systems are invariant under linear transformation $\bar{x} = P(t)x$, where $P(t)$ is nonsingular and continuously differentiable for all $t \in (-\infty, \infty)$. . . . .(4p)

Solution: Under the new coordinates, we have $\dot{\bar{x}} = \bar{A}(t)\bar{x} + \bar{B}(t)u$, $y = \bar{C}(t)\bar{x}$, where $\bar{A}(t) = P(t)A(t)P^{-1}(t) + \dot{P}(t)P^{-1}(t)$, $\bar{B}(t) = P(t)B(t)$, $\bar{C}(t) = C(t)P^{-1}(t)$. We can easily show that $\bar{\Phi}(t,s) = P(t)\Phi(t,s)P^{-1}(s)$ and the rest follows using the respective Gramian matrix.

4. Consider the inverted pendulum as we did in the lecture (see Figure 1). The following equation describes the motion of the pendulum around the equilibrium $\theta = 0,\ \dot{\theta} = 0$:

$$L\ddot{\theta} - g\sin(\theta) + \ddot{x}\cos(\theta) = 0$$

(a) We consider $\ddot{x}$ as the input u and θ as the output y. Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. Derive the state space model for the linearized system (i.e. let $\sin(\theta) \approx \theta$ and $\cos(\theta) \approx 1$), and show that the model you derive is both controllable and observable. ....(2p)

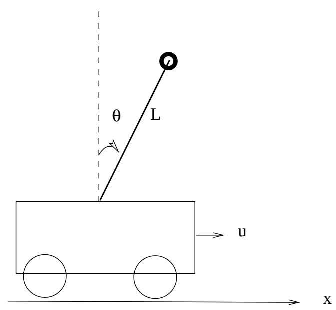
(b) Setting $u(t) = 0$ and using the linearized model, can we find an initial state $x_1(0) \neq 0$ and $x_2(0)$ such that $x_1(t) = 0$ for all $t \geq T$ where $T > 0$ is some finite time? ....(2p)

Solution: omitted

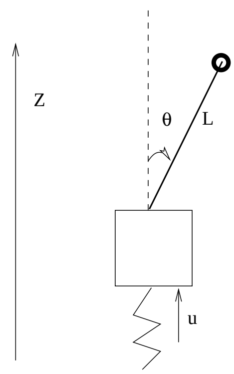
5. Now consider an inverted pendulum with oscillatory base (see Figure 2). The following equation describes the motion of the pendulum around the equilibrium $\theta = 0,\ \dot{\theta} = 0$:

$$L\ddot{\theta} - g\sin(\theta) - \ddot{z}\sin(\theta) = 0$$

(a) We consider $\ddot{z}$ as the input u and θ as the output y. Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. Derive the state space model for the *linearized* system. ....(1p)



Figur 1: Inverted pendulum on a cart.



Figur 2: Pendulum with oscillatory base.

(b) Is the model you derive in (a) controllable? .... (1p)

(c) Let $z = 0.1\sin(\omega t)$ (be careful with the variable!). Use Matlab simulation to find out what will happen to the motion (use the original nonlinear model!) of the pendulum near $\theta = 0$ when the oscillation is "slow" and when the oscillation is "fast". Take $L = 1,\ \theta(0) = 0.02$ and $\dot{\theta}(0) = 0$. .... (3p)

Solution: omitted