Math. Systems Theory 18, 293–294 (1985)

Mathematical Systems Theory

Editor's Introduction to Euler's "An Essay on Continued Fractions" translated by B. F. Wyman and M. F. Wyman

Christopher Byrnes

Department of Mathematics, Arizona State University, Tempe, AZ 84287

We at *Mathematical Systems Theory* are pleased to publish a translation, by B. F. Wyman and M. F. Wyman of Euler's 1737 treatise, "De fractionibus continuis dissertatio," on continued fractions. Our reason for having encouraged this translation is not the near coincidence of the anniversaries of Euler's birth and death (1707–1783) nor the desire to see all classics translated into a modern tongue (very few of Euler's works have been translated from Latin), but because this particular treatise is of central importance in an area of current research interest in mathematical systems theory.

The relevance of this paper to the "partial realization" problem and its relation to the Riccati equation were apparently first discovered during a seminar on algebraic system theory at the Universität Bremen during the Fall of 1982. Since that time, knowledge of this paper and of the interesting connections it raises has been confined to a growing, but small, group of algebraic system theorists. It is our intention, thanks to the erudite translation by the Wymans, to make this available to the entire community.

Euler's starting point is the very basic question:

Is e rational or irrational?

Appreciating as everyone does the fact that a real number α is rational if, and only if, the continued fraction expansion

$$\alpha = n + \cfrac{1}{\alpha_0 + \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_3 + \text{etc.}}}}}$$

($n, \alpha_i \in \mathbf{Z}$) terminates, Euler intended to show that e is irrational by showing that for $\alpha = e$, one has $n = 2$ and

$$[\alpha_0, \alpha_1, \alpha_2, \ldots] = [1,2,1,1,4,1,1,6,\ldots] \tag{1}$$

While everyone who enjoys mathematics is certainly interested in the question, rationality is a central concept in system theory and for this reason, Euler's techniques are of special interest to our audience.

Briefly, Euler proposes to prove (1) by more generally expressing the function $e^{1/z}$ in a continued fraction expansion, obtaining functions $\alpha_i(z)$. If the $\alpha_i(z)$ are known, then one could recover (1) by simply evaluating the $\alpha_i(z)$ at $z = 1$. Euler then, in a stroke of inspiration, finds the $\alpha_i(z)$ by solving a differential equation for the $\alpha_i(z)$. This equation is easier to express in terms of the partial ratios, i.e., the rational functions

$$p_n(z)/q_n(z)$$

obtained by summing the partial continued fraction expansion, where one sets

$$\alpha_{n+j}(z) = 0, \qquad j \geqslant 1.$$

Indeed, the p_i and q_i satisfy second order linear equations, similar to the recurrence relations due to Chebychev and others. Euler deduces the same for their "limits" p_∞, q_∞ and from this realizes that, of course, the ratio satisfies a Riccati equation. This he solves explicitly, obtaining (1) by setting $z = 1$.

Quite remarkably, Euler's treatise contains many of the standard tools of partial realization theory—indeed, Euler computes the partial realizations of the nonrational function $e^{1/z}$. Thus, the reader will find forerunners of continued fraction expansions of nonrational functions and their relation to the Euclidean algorithm (sometimes believed to have been discovered only recently), Padé approximants of the same, and the well-known recurrence relations attributed to Chebychev and to Stieltjes. To be sure, Euler did not systematically classify these connections, but he can still add one exciting and new ingredient—the deep connection between continued fractions and the Riccati equation—as a conceptual and a computational tool, especially when combined with state-space methods. It is on these bases that we present a translation of Euler's paper in the context of mathematical system theory.