Causal Wiener filtering

Assume that x and y are two jointly stationary second order processes, with spectral density given by

$$\Phi(z) = \begin{bmatrix} \Phi_x & \Phi_{xy} \\ \Phi_{yx} & \Phi_y \end{bmatrix} (z)$$

We would like to find a strictly causal function $F(t)$ such that $\chi(t) = \hat{F}(z)y(t)$ is the best linear estimator of $x(t)$ given $y(t-1), y(t-2), \cdots$.

If $y(t)$ is a white noise process, it is easy to see that the optimal filter is given by a causal function

$$\hat{F}(z) = \Big[\Phi_{xy}(z)\Big]_+ ,$$

where $[\cdot]_+$ denotes the strictly causal part of the function.

Lemma 4.1.4. *Assume the observation process y is normalized white noise. Then the matrix function F defining the best linear causal estimator $\chi(t)$ of $x(t)$ given the past history of y up to and including time $t-1$, i.e.*

$$\chi(t) := \mathrm{E}[x(t)|\mathbf{H}_t^-(y)] = \sum_{s=-\infty}^{\infty} F(t-s)y(s)$$

is given by

$$F(t) = \begin{cases} \Lambda_{xy}(t), & t > 0 \\ 0, & t \leq 0 \end{cases}$$

where Λ_{xy} is the cross covariance matrix of the processes x and y

Proof: The orthogonality condition provides

$$\mathrm{E}\{(x(t) - \chi(t))y(\tau)'\} = \mathrm{E}\{(x(t) - \sum_{s=-\infty}^{\infty} F(t-s)y(s))y(\tau)'\} = 0, \quad \tau \leq t-1.$$

which can be written as

$$\Lambda_{xy}(t-\tau) = \sum_{s=-\infty}^{\infty} F(t-s)I\delta(t-s) = F(t-\tau), \quad \tau \leq t-1.$$

And in order that $\chi(t) \in \mathbf{H}_t^-(y)$, the function F has to be strictly causal. $\Box$

In general, the process y is not white noise, and then the following trick using a whitening filter can be applied. Assuming that we can determine a minimum-phase spectral factor W of Φ_y, the process $e(t) = W^{-1}(z)y(t)$

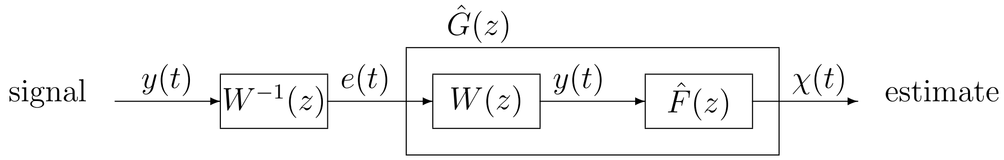

Figure 1: Cascade structure of the Wiener filter

is the (forward) normalized innovation process of $y(t)$, i.e. a white noise process and we can consider the cascade form of the Wiener filter illustrated in Figure 1.

The part $\hat{G}$ can now be determined as $\hat{F}$ above, the only difference is that we have to exchange Φ_{xy} with

$$\Phi_{xe} = \Phi_{xy} W^{-*}. \tag{1}$$

The estimator is then given by

$$\chi(t) = \frac{1}{W(z)} \left[\frac{\Phi_{xy}(z)}{W^*(z)} \right]_+ y(t).$$

To derive the expression (1), assume that $\epsilon(t)$ is the innovation process of the joint process $(x(t), y(t))$. Then consider the spectral representations

$$x(t) = \int_{-\pi}^{\pi} e^{it\theta} d\hat{x}(\theta) = \int_{-\pi}^{\pi} e^{it\theta} W_x(e^{i\theta}) d\hat{\epsilon}(\theta).$$

and

$$y(t) = \int_{-\pi}^{\pi} e^{it\theta} d\hat{y}(\theta) = \int_{-\pi}^{\pi} e^{it\theta} W_y(e^{i\theta}) d\hat{\epsilon}(\theta),$$

The cross-covariance $\Lambda_{xy}(\tau) = \mathrm{E}\{x(t+\tau)y(t)\}$ can be written

$$\begin{aligned} \Lambda_{xy}(\tau) &= \mathrm{E}\left\{ \int_{-\pi}^{\pi} e^{i(t+\tau)\theta} W_x(e^{i\theta}) d\epsilon(\theta) \int_{-\pi}^{\pi} e^{-it\theta} W_y(e^{-i\theta}) d\epsilon(\theta) \right\} \\ &= \int_{-\pi}^{\pi} e^{i\tau\theta} W_x(e^{i\theta}) W_y(e^{-i\theta}) \frac{d\theta}{2\pi}, \end{aligned}$$

and thus $\Phi_{xy} = W_x W_y^*$. Now, since $e(t) = W^{-1}(z)y(t) = W^{-1}(z)W_y(z)\epsilon(t)$ it holds that

$$\Phi_{xe} = W_x (W^{-1} W_y)^* = W_x W_y^* W^{-*} = \Phi_{xy} W^{-*},$$

which proves (1).

Example 1

Assume that

$$\Phi_y(z) = \frac{5/4 + 1/2(z + z^{-1})}{5/4 + 1/2(z + z^{-1})}, \quad \Phi_{xy}(z) = \frac{z+\alpha}{z+\beta}$$

Then, it is easy to see that

$$\Phi_y(z) = \frac{(z + 1/2)(z^{-1} + 1/2)}{(z - 1/2)(z^{-1} - 1/2)} = W(z)W(z^{-1}),$$

where

$$W(z) = \frac{z + 1/2}{z - 1/2}$$

is stable and minimum-phase.

Then, assuming $\beta \neq 2$ (simple poles of the denominator below)

$$\frac{\Phi_{xy}(z)}{W(z^{-1})} = \frac{z+\alpha}{z+\beta}\frac{z^{-1} - 1/2}{z^{-1} + 1/2} = \frac{A}{z+\beta} + \frac{B}{z^{-1} + 1/2} + C,$$

where

$$A = (\alpha - \beta)\frac{1 + \beta/2}{1 - \beta/2}, \quad B = \frac{1 - \alpha/2}{1 - \beta/2}, \quad C = \frac{1 - \alpha + \beta/2}{1 - \beta/2}.$$

Now, assuming $|\beta| < 1$

$$\left[\frac{\Phi_{xy}(z)}{W(z^{-1})}\right]_+ = \frac{A}{z+\beta},$$

and

$$\frac{1}{W(z)}\left[\frac{\Phi_{xy}(z)}{W(z^{-1})}\right]_+ = \frac{z - 1/2}{z + 1/2}\frac{A}{z+\beta},$$

Example 2

Consider now a pair (x, y) of jointly stationary stochastic processes, with spectral density given by

$$\Phi(z) = \begin{bmatrix} \Phi_x & \Phi_{xy} \\ \Phi_{yx} & \Phi_y \end{bmatrix} = \begin{bmatrix} \frac{200+40(z+z^{-1})}{3-(z+z^{-1})} & \frac{1}{z+1/2} \\ \frac{1}{z^{-1}+1/2} & e^{z+z^{-1}} \end{bmatrix}.$$

The stable minimum phase spectral factor W of Φ_y is given by

$$W(z) = e^{z^{-1}} = \sum_{k=0}^{\infty} \frac{z^{-k}}{k!} \in H^2.$$

It is easy to see that also $W^{-1} \in H^2$.

Then

$$\hat{G}(z) = \left[\frac{\Phi_{xy}(z)}{W(z^{-1})}\right]_+ = \left[\frac{\frac{1}{z+1/2}}{e^z}\right]_+ = \left[\frac{e^{-z}}{z+1/2}\right]_+,$$

and the causal part can be determined as follows

$$\begin{aligned}
\hat{G}(z) &= e^{1/2}\left[\frac{e^{-(z+1/2)}}{z+1/2}\right]_+ \\
&= e^{1/2}\left[\frac{1-(z+1/2)+\frac{1}{2!}(z+1/2)^2-\frac{1}{3!}(z+1/2)^3+\cdots}{z+1/2}\right]_+ \\
&= e^{1/2}\left(\left[\frac{1}{z+1/2}\right]_+ - [1]_+ + \left[\frac{1}{2!}(z+1/2)\right]_+ - \left[\frac{1}{3!}(z+1/2)^2\right]_+ + \cdots\right) \\
&= e^{1/2}\frac{1}{z+1/2}.
\end{aligned}$$

Alternatively, note that $G(t) = 0$ for $t \le 0$, and for $t > 0$

$$G(t) = \Lambda_{xe}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{it\theta}\frac{\Phi_{xy}(e^{i\theta})}{W(e^{-i\theta})}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{it\theta}\frac{e^{-e^{i\theta}}}{e^{-i\theta}+1/2}d\theta.$$

By using the Residue theorem

$$G(t) = z^{t-1}e^{-z}\big|_{z=-1/2} = \left(-\frac{1}{2}\right)^{t-1} e^{1/2}, \quad t = 1, 2, \cdots.$$

Then

$$\hat{G}(z) = \sum_{t=1}^{\infty}\left(-\frac{1}{2}\right)^{t-1} e^{1/2}z^{-t} = e^{1/2}z^{-1}(1+1/2z^{-1})^{-1} = \frac{e^{1/2}}{z+1/2},$$

as before.

The optimal prediction is given by the filter

$$\frac{1}{W(z)}\left[\frac{\Phi_{xy}(z)}{W(z^{-1})}\right]_+ = \frac{1}{e^{z^{-1}}}\frac{e^{1/2}}{z+1/2} = \frac{e^{1/2-z^{-1}}}{z+1/2}.$$