Portfolio Risk Measures over Time

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Abstract

The main purpose of this master thesis, commissioned by Handelsbanken, is to provide a framework for estimation of Value-at-Risk in fixed income portfolios over time-horizons exceeding one day. We conduct a comparative study of various methods of scaling one day Value-at-Risk and their respective performance, focusing on time-horizons of ten days, three months and one year, taking into consideration the specific challenges of fixed income asset classes. We also investigate Value-at-Risk scaling in the setting of non-stationary portfolio weights, concentrating on the dynamic hedging strategy CPPI.

We find that the square-root-of-time rule performs relatively well in a majority of cases, both on simulated data and on real fixed income index data. Combined with other methods such as bootstrap, and using a two-step method, scaling Value-at-Risk to three months and one year works well for low volatility assets, less so for more volatile assets with large jumps in price.

The implementation of CPPI show an expected change in the distribution of the returns where we see less large drops in the portfolio value accompanied by a decrease in the expected return; however we also see an increase in complexity and assumptions which one has to address when deciding if the implementation of a so called Management Action structure is appropriate..

Acknowledgements

We would like thank Harald Lang at KTH and Joakim Jansson and Magnus Lundin at Handelsbanken for their invaluable help, guidance and support.

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Joakim Ahlinder, Magnus Hanson

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1. Introduction

1.1 Motivation

Value-at-Risk has been the dominating way of measuring risk in applied risk management, although it has taken substantial criticism from academics. One of the reasons for its widespread use within financial institutions is The Basel Committee on Banking Supervision's recommendation that the capital reserves of these companies' portfolios should be proportional to the 10-day Value-at-Risk at the 99% confidence level. The problem that arises in estimating this figure is the lack of sufficient 10-day return data. The way around this recommended in Basel II is to scale 1-day Value-at-Risk with the square-root-of-time rule. The satisfactory performance when applied to a 10-day horizon is documented by several authors. However, the suitability of using the square-root-of-time rule for longer time horizons, for example when calculating yearly risk measures (e.g. Economic Capital, EC) for banks or the measure of risk in trading books recently recommended by the committee, IRC (Incremental Risk Charge, 1 year 99.9% VaR), is less investigated. Also, focus has been on equity rather than fixed income. A significant part of this study will therefore address the issues arising with fixed income asset classes and the scaling of Value-at-Risk to time periods exceeding 10 days.

Recent updates in the insurance equivalent of Basel II, Solvency II, has allowed for taking into account the effects of non-stationary portfolio weights in the form of so called Management Actions in the internal risk models of insurance companies. We investigate the impact on 3-month Value-at-Risk when changing from a stationary portfolio to one which is rebalanced each month and we will exemplify this by implementing the hedging strategy CPPI (Constant Proportion Portfolio Insurance).

1.2 Problem formulation

The purpose of this master thesis is to provide a framework for estimation of Value-at-Risk in fixed income portfolios over time-horizons exceeding one day. We will conduct a comparative study of various methods of scaling one day Value-at-Risk and their respective performance, focusing on time-horizons of ten days, three months and one year, taking into consideration the specific challenges of fixed income asset classes. We will also investigate the effect of non-stationary portfolio weights on portfolio risk, focusing on the dynamic hedging strategy CPPI.

1.3 Methodology

The main focus of this study is to compare methods for the scaling of Value-at-Risk to time-horizons exceeding one day. To accomplish this we begin with a thorough analysis of a number of fixed income indices to be used for this purpose, since understanding the statistical properties of the data set is crucial in drawing further conclusions. We analyze the data in two steps. To begin with we look at histograms and descriptive statistics in chapter 3. We then proceed with a distribution analysis in chapter 4, were we investigate the data tails and the amount of autocorrelation and heteroskedasticity. Finally we attempt to estimate parameters for the respective data sets for AR, GARCH and AR-GARCH models.

In chapter 5 we conduct a comparative study of several methods of scaling 1-day Value-at-Risk to 10-day Value-at-Risk. The study is based on data simulated from the distributions fitted in chapter 4. In Chapter 6 we focus on 3-month and 1-year Value-at-Risk for real fixed income data and the special considerations to be taken for longer time horizons. We then perform back testing analysis to investigate the precision of the different procedures. These results are then compared to those in chapter 5.

In chapter 7 we investigate the effects of non-stationary portfolio weights. We do this by comparing a risk model estimating quarterly Value-at-Risk with stationary portfolio weights to one that is rebalanced each month in accordance with the so called CPPI structure.

Chapter 8 finally consists of conclusions regarding the most appropriate ways of scaling Value-at-Risk in fixed income portfolios and considerations to be taken when estimating risk in a non-stationary portfolio, as well as suggestions for future research.

2. Theoretical background

2.1 Price process models

For a given daily price process P_t we will in this study consider the logged returns, i.e.

$$X_t = \log\left(\frac{P_t}{P_{t-1}}\right)$$

This then means that the N-day log-returns, denoted Z_t^N, are the arithmetic sum of the 1-day log-returns:

$$Z_t^N = \log\left(\frac{P_t}{P_{t-N}}\right) = X_t + X_{t-1} + \cdots + X_{t-(N-1)}$$

Common returns, i.e. $R_t = (P_t - P_{t-1})/P_{t-1}$ *, are related to logged returns in the following as* $R_t = e^{X_t} - 1$

2.2 Value-at-Risk (VaR)

For a random variable R, which is considered the return of some asset, the Value-at-Risk at level α of R is defined as

$$VaR_\alpha(R) = -\inf\{x \in \mathbb{R} | \mathbb{P}[R \leq x] \geq 1-\alpha\}$$

$$or \;\; VaR_\alpha(R) = \sup\{x \in \mathbb{R} | \mathbb{P}[R \leq -x] \geq 1-\alpha\}$$

i.e. $VaR_\alpha(R)$ *is the negative* $(1-\alpha)$*-quantile of R*

Using logged returns (X) we have the relation

$$VaR_\alpha(R) = 1 - \exp(-VaR_\alpha(X))$$

2.3 Random Walk

A series of independent innovations, commonly used for modeling logged returns:

$$X_t = \sigma\epsilon_t, \;\; \mathbb{E}[\epsilon_t] = 0, \qquad \mathbb{E}[\epsilon^2{}_t] = 1, \qquad \epsilon_t \; i.i.d.$$

In this study we use normal and student-t distributed innovations ($\boldsymbol{\epsilon_t}$ *)*

2.4 AR(1)

An Autoregressive process X_t of order (lag) 1, AR(1), is defined as

$$X_t = c + \varphi X_{t-1} + \boldsymbol{\epsilon_t}, \quad \epsilon_t \; iid \sim \mathcal{N}(0,1)$$

2.5 GARCH(1,1)

A Generalized Autoregressive Conditional Heteroskedasticity process X_t of order (1,1), GARCH(1,1) , is defined as

$$X_t = \sigma_t \epsilon_t, \qquad \epsilon_t \; iid \sim \mathcal{N}(0,1)$$
$$\sigma_t^2 = a_0 + a X_{t-1}^2 + b\sigma_{t-1}^2$$

2.6 AR(1)-GARCH(1,1)

A generalization of the GARCH(1,1) model, the AR(1)-GARCH(1,1) process, is defined as

$$X_t = \mu_t + \sigma_t \epsilon_t, \qquad \epsilon_t \; iid \sim \mathcal{N}(0,1)$$
$$\mu_t = \varphi X_{t-1}$$
$$\sigma_t^2 = a_0 + a(X_{t-1} - \mu_{t-1})^2 + b\sigma_{t-1}^2$$

We may also assume the innovations ϵ_t to be e.g. Student-t distributed instead of normally distributed as in 2.3-2.6.

3. Data

In order to choose appropriate models for estimating the Value-at-Risk we start by studying a set of fixed income indices to understand the distribution and behavior of fixed income securities. The indices are the following: the FTSE Euro Corporate Bonds All Maturities (Bloomberg ticker FECVC), Dow Jones Corporate Bond Total Return Index (Bloomberg ticker DJCBT), JPMorgan Emerging Markets Bonds Index (Bloomberg ticker JPGCCOMP), Credit Suisse High Yield Index II Total Return (DLJHTR), the index of the Swedish Nation Debt Office's and the mortgage institutions borrowings via bonds, OMRX-Bond Index (Bloomberg ticker RXBO).

FTSE Euro Corp. Bond Index is denoted in euro and consists of investable and liquid investment grade bonds. Dow Jones Corp. Bond Index is denoted in US dollars and consists of readily tradable, high grade bonds with equal weighting and monthly rebalancing. Credit Suisse High Yield Index contains US dollar denoted bonds with a rating lower than investment grade. The index JPMorgan Em. Mark. Bond Index consists of US dollar denoted emerging market bonds, weighted by market-capitalization.

The data were provided by Bloomberg and includes time series of varying length and incremental period. All series are daily apart from Credit Suisse High Yield Index which is weekly. We examine the distribution of the time series for different lengths of time and we start by examining each time series for the full length available (which means time series of a length of upwards sixteen years). However it becomes hard to justify stationarity and consistency in parameter estimation when using such long time series, especially in the light of the recent financial crisis and its impact on the returns. We therefore continue our analysis using time series stretching one year back in time as well as two years back in time. This gives us a range of reasonable estimates on, for example, degrees of freedom for the student-t distributions and the parameters for the different time series models. The standard deviation of daily log-returns ranges from 0.0072 for JPMorgan Emerging Market Bonds Index to 0.0016 for the less volatile OMRX-Bond Index.

Data set	Nr. Years	STD daily returns
Dow Jones Corp. Bond Index	3	0.0049
Credit Suisse High Yield Index	13	0.0092 (weekly)
FTSE Euro Corp. Bond Index	4	0.0016
JPMorgan Em. Mark. Bond Index	15	0.0072
OMRX-Bond Index	13	0.0016

Dow Jones Corporate Bond Total Return Index 2006-11-28 – 2009-10-06

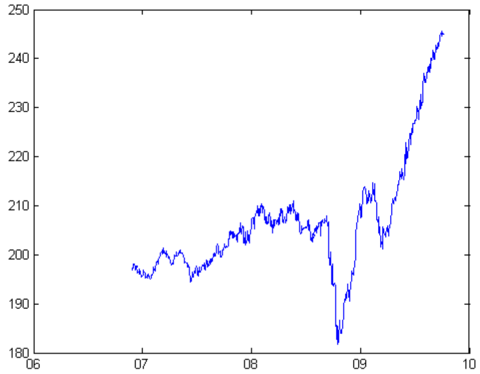

Figure 3.1: Price development Dow Jones Corp. Bond Index

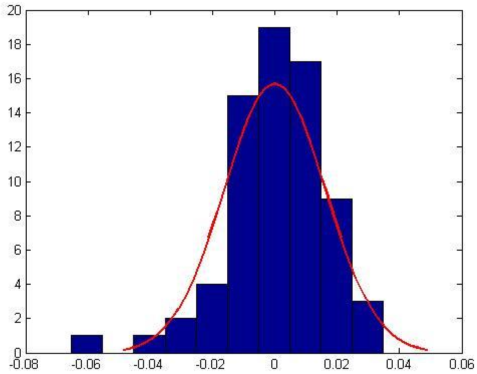

Figure 3.2: Histogram of monthly Dow Jones Corp. Bond Index returns vs. normal distribution

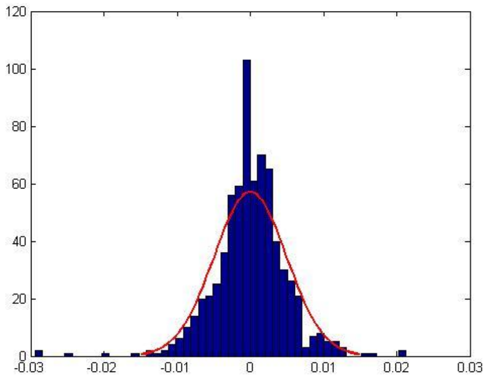

Figure 3.3: Histogram of daily returns vs. normal distribution Dow Jones Corp. Bond Index

Dow Jones Corp. Bond Index Descriptive Statistics	
Mean (daily returns)	3.04e-4
Standard deviation (daily returns)	0.0049
Skewness (daily returns)	-0.5696
Kurtosis (daily returns)	8.2368
Mean (monthly returns)	0.0006
Standard deviation (monthly returns)	0.0248
Skewness (monthly returns)	-0.1150
Kurtosis (monthly returns)	4.4100

Table 3.1: Descriptive statistics for daily and monthly returns Dow Jones Corp. Bond Index

Credit Suisse High Yield Index II Total Return 1996-07-11 - 2009-10-01

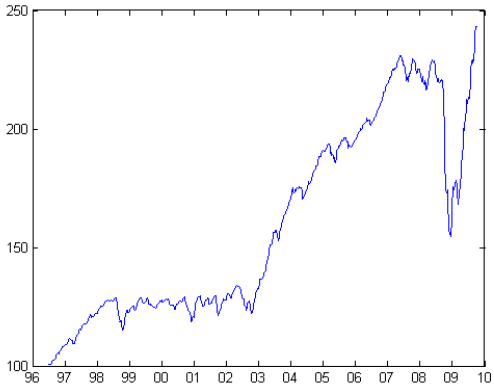

Figure 3.4: Price development Credit Suisse High Yield Index

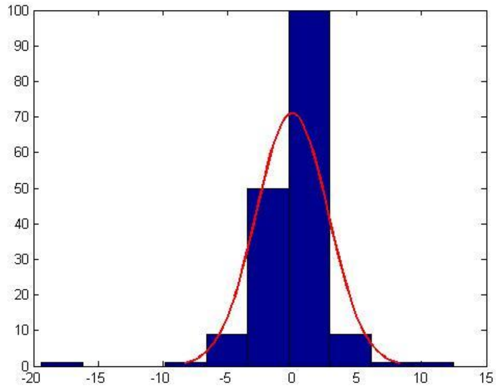

Figure 3.5: Histogram of monthly log-returns vs. normal distribution Credit Suisse High Yield Index

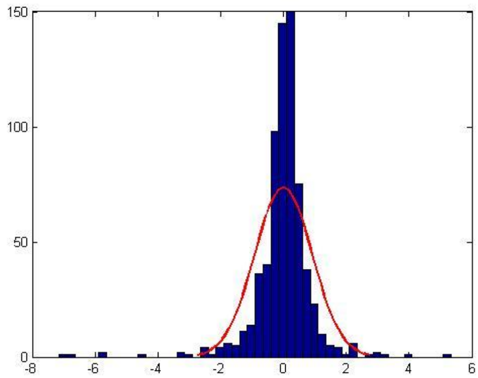

Figure 3.6: Histogram of weekly returns vs. normal distribution Credit Suisse High Yield Index

Credit Suisse High Yield Index Descriptive Statistics	
Mean (weekly returns)	0.0013
Standard deviation (weekly returns)	0.0092
Skewness (weekly returns)	-2.0313
Kurtosis (weekly returns)	20.8702
Mean (monthly returns)	0.0049
Standard deviation (monthly returns)	0.0259
Skewness (monthly returns)	-2.1454
Kurtosis (monthly returns)	15.8232

Table 3.2: Descriptive statistics for weekly and monthly returns Credit Suisse High Yield Index

FTSE Euro Corporate Bonds All Maturities Price 2005-04-26 - 2009-10-06

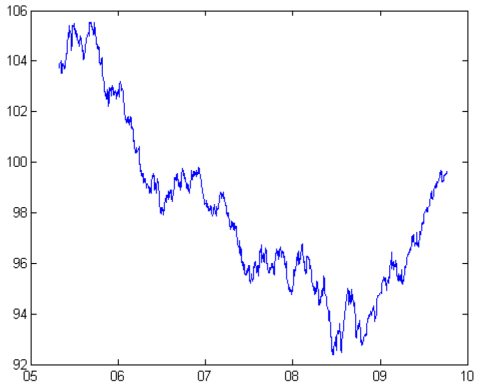

Figure 3.7: Price development FTSE Euro Corp. Bond Index

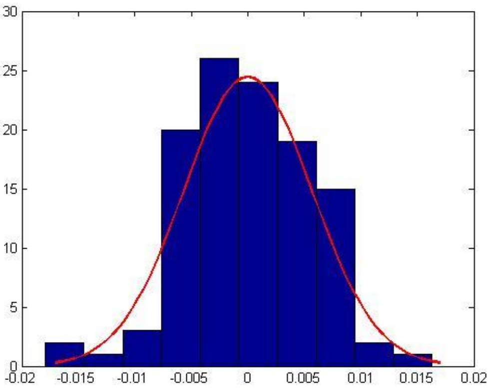

Figure 3.8: Histogram of monthly log- returns vs. normal distribution FTSE Euro Corp. Bond Index

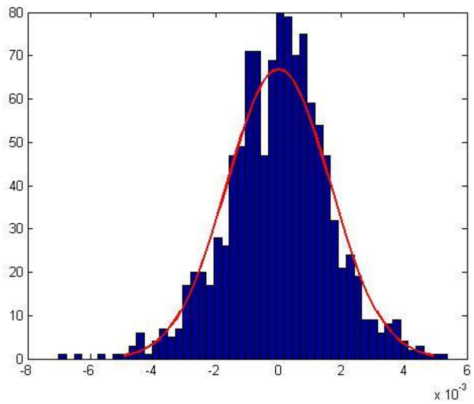

Figure 3.9: Histogram of daily returns vs. normal distribution FTSE Euro Corp. Bond Index

FTSE Euro Corp. Bond Index Descriptive Statistics	
Mean (daily returns)	-3.63e-5
Standard deviation (daily returns)	0.0016
Skewness (daily returns)	-0.1821
Kurtosis (daily returns)	3.7442
Mean (monthly returns)	-7.74e-4
Standard deviation (monthly returns)	0.0086
Skewness (monthly returns)	-0.3872
Kurtosis (monthly returns)	2.6179

Table 3.3: Descriptive statistics for daily and monthly returns FTSE Euro Corp. Bond Index

JPMorgan Emerging Markets Bonds Index EMBI Global Diversified Composite 1994-01-03 - 2009-10-06

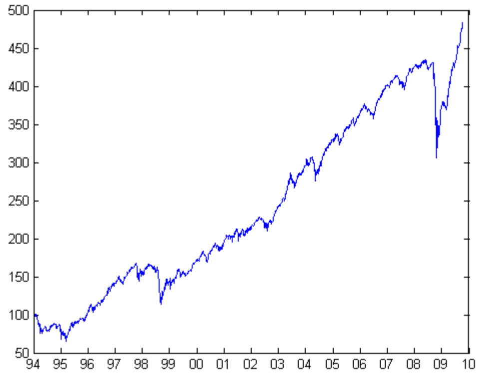

Figure 3.10: Price development JPMorgan Em. Mark. Bond Index

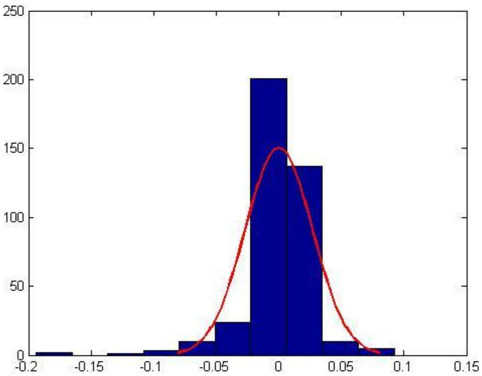

Figure 3.11: Histogram of monthly log-returns vs. normal distribution JPMorgan Em. Mark. Bond Index

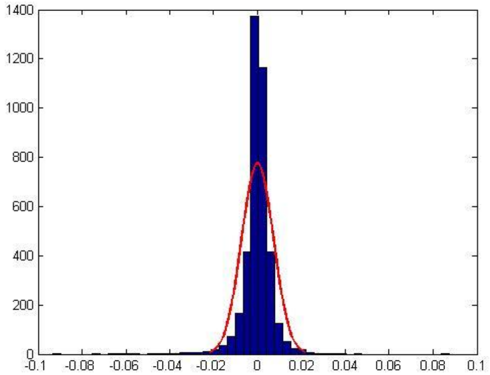

Figure 3.12: Histogram of daily returns vs. normal distribution JPMorgan Em. Mark.
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JPMorgan Em. Mark. Bond Index Descriptive Statistics	
Mean (daily returns)	4.01e-4
Standard deviation (daily returns)	0.0072
Skewness (daily returns)	-1.6494
Kurtosis (daily returns)	32.1203
Mean (monthly returns)	0.0078
Standard deviation (monthly returns)	0.0424
Skewness (monthly returns)	-2.4830
Kurtosis (monthly returns)	16.5893

Table 3.4: Descriptive statistics for daily and monthly returns JPMorgan Em. Mark. Bond Index .

OMRX-Bond, Swedish Nation Debt Office's and the mortgage institutions borrowings via bonds 1996-09-24 - 2009-09-24

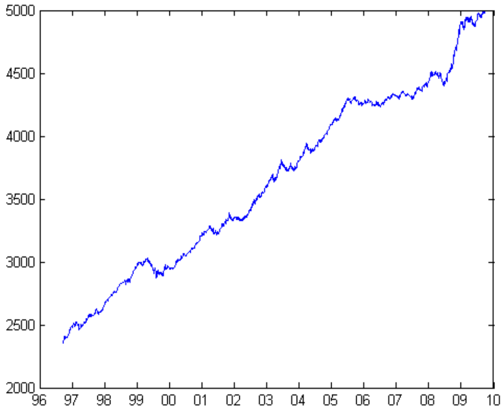


Figure 3.13: Price development OMRX-Bond Index

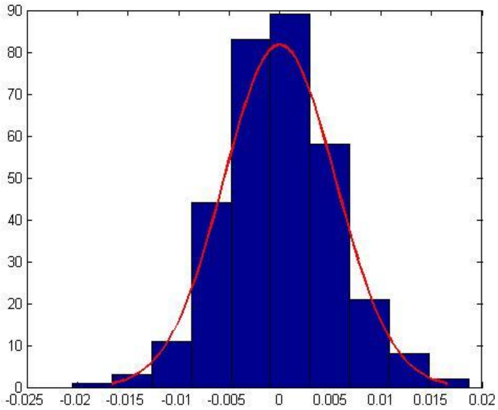


Figure 3.14: Histogram of monthly log-returns vs. normal distribution OMRX-Bond Index

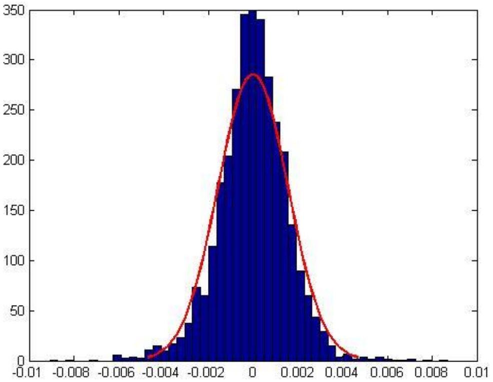


Figure 3.15: Histogram of daily returns vs. normal distribution OMRX-Bond Index

OMRX-Bond Index Descriptive Statistics	
Mean (daily returns)	2.34e-4
Standard deviation (daily returns)	0.0016
Skewness (daily returns)	-0.1569
Kurtosis (daily returns)	5.4544
Mean (monthly returns)	0.0047
Standard deviation (monthly returns)	0.0081
Skewness (monthly returns)	0.1130
Kurtosis (monthly returns)	3.3465

Table 3.5: Descriptive statistics for daily and monthly returns OMRX-Bond Index

4. Distribution Analysis

In this chapter we do a distribution analysis on the different data sets described in chapter 3. The analysis will be conducted on 1 and 20 day (monthly) log-returns and will be based both on an IID assumption with a random walk model, and time series analysis to investigate autocorrelation and heteroskedasticity. In the case of fitting random walk parameters with normal and student-t distributed innovations we use QQ-plots of both 1 and 20 day log-returns to estimate their respective parameters. We then turn to investigating autocorrelation and heteroskedasticity with autocorrelation plots and finally modeling the return data with stochastic processes commonly used to model financial returns (AR, GARCH, AR-GARCH).

The purpose of this chapter is to get a feel for how different classes of fixed income assets tend to be distributed, in order to create a framework for scaling their respective value-at-risk.

4.1 Quantile-Quantile Plotting

We limit ourselves to comparing the quantiles of the data to the quantiles of the normal and the student-t distribution with so called qq-plots (quantile-quantile plots) and then proceed with estimation of the parameters of these distributions from the plots, in order to understand the tail-behavior of our data sets.

If we define the ordered sample as

$$X_{n,n} \leq X_{n-1,n} \leq \cdots \leq X_{1,n}$$

then the qq-plot consists of the points

$$\{(X_{k,n}, F^{\leftarrow}(\tfrac{n-k+1}{n+1})): k = 1, \dots, n\}$$

A linear plot implies that the reference distribution is approximately a linear transformation of the data distribution, i.e. from the same location-scale family.

We compared both daily and monthly data to normal distributions as well as student-t distributions with degrees of freedom ranging from 2 to 12 for all data sets. In figures 4.1-4.10 below shows the most appropriate student-t distribution versus the standard normal distribution for the respective data sets. An S-shaped curve, i.e. where the top of the curve shifts to the right and the bottom of the curve to the left of the straight line, is an implication of the sample distribution (x-axis) having more mass in the tails (top and bottom quantiles) than the reference distribution, whereas an inverted S-shape where the top of the curve shifts to the left and the bottom to the right implies that the sample distribution has lighter tails than the reference distribution. The more linear plot the better the different quantiles of the reference distribution fit the data. This is however just a rough method to investigate the tail behavior of the data, and the fit will not be perfect since there is no account taken for dependence etc.

OMRX-Bond Index

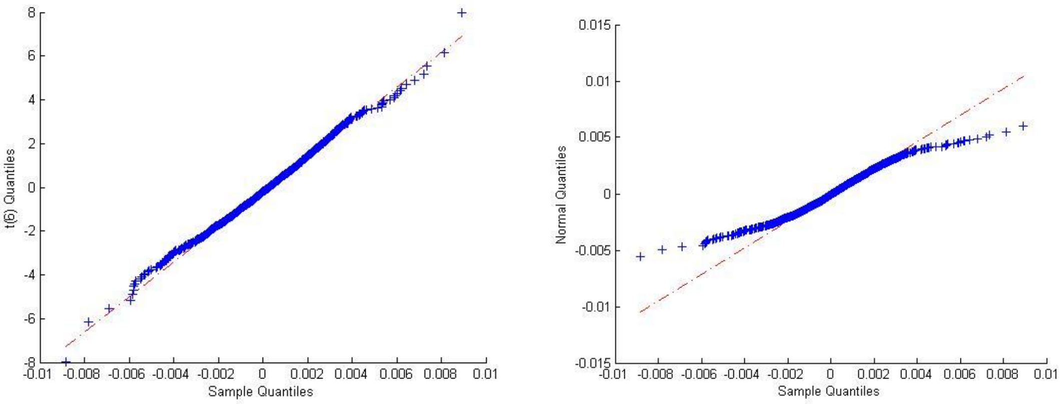


Figure 4.1.1: OMRX-Bond Index daily log-return quantiles vs. t(6) (left) and normal(0,1) (right).

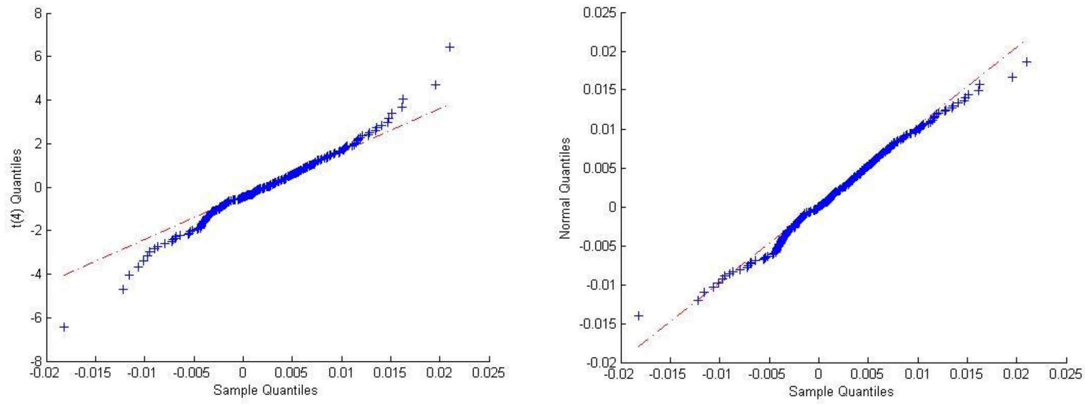


Figure 4.1.2: OMRX-Bond Index monthly log-return quantiles vs. t(4) (left) and normal(0,1) (right).

FTSE Euro Corp. Bond Index

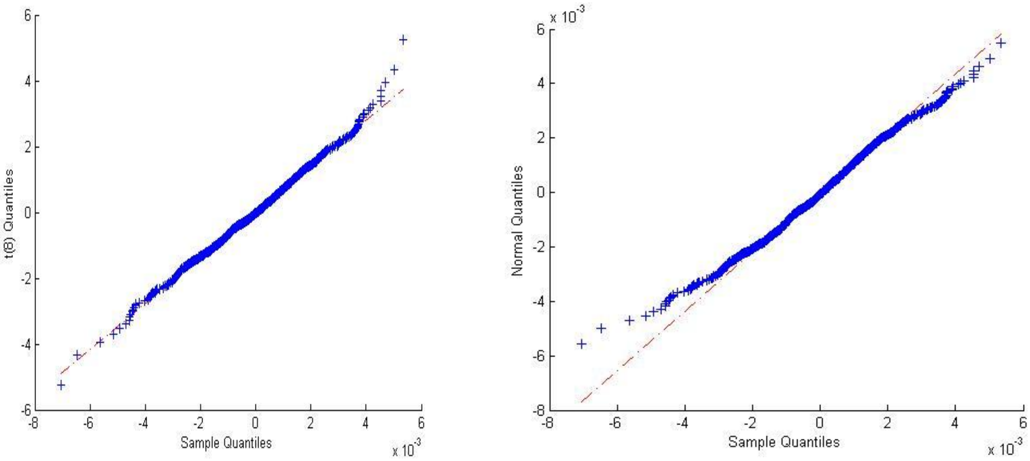


Figure 4.1.3: FTSE Euro Corp. Bond Index daily log-return quantiles vs. t(8) (left) and normal(0,1) (right).

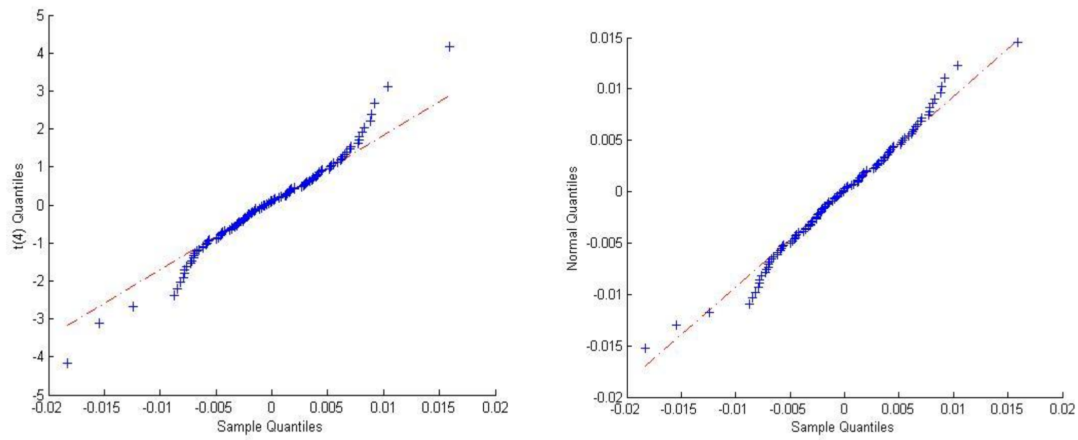


Figure 4.1.4: FTSE Euro Corp. Bond Index monthly log-return quantiles vs. t(4) (left) and normal(0,1) (right).

Dow Jones Corp. Bond Index

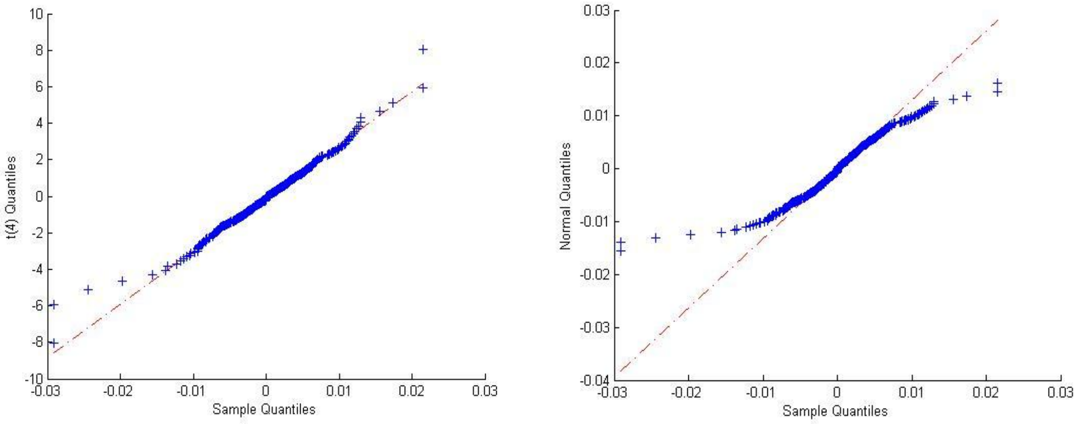


Figure 4.1.5: Dow Jones Corp. Bond Index daily log-return quantiles vs. t(4) (left) and normal(0,1) (right).

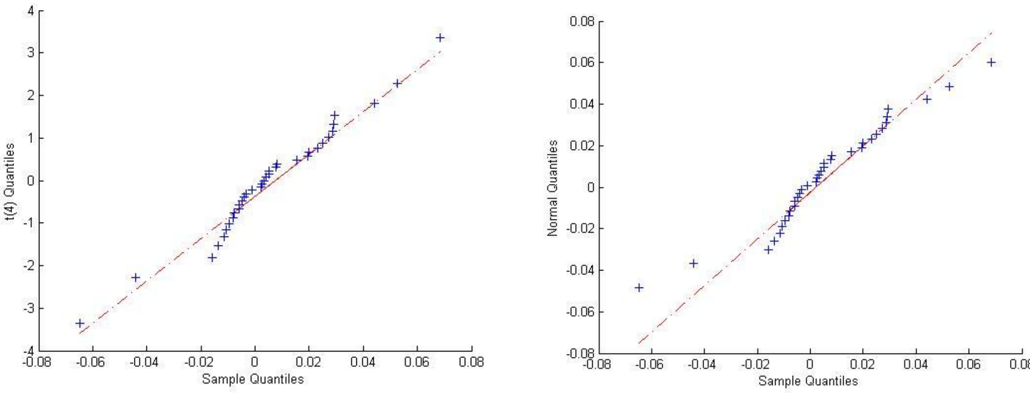


Figure 4.1.6: Dow Jones Corp. Bond Index monthly log-return quantiles vs. t(4) (left) and normal(0,1) (right).

JPMorgan Em. Mark. Bond Index

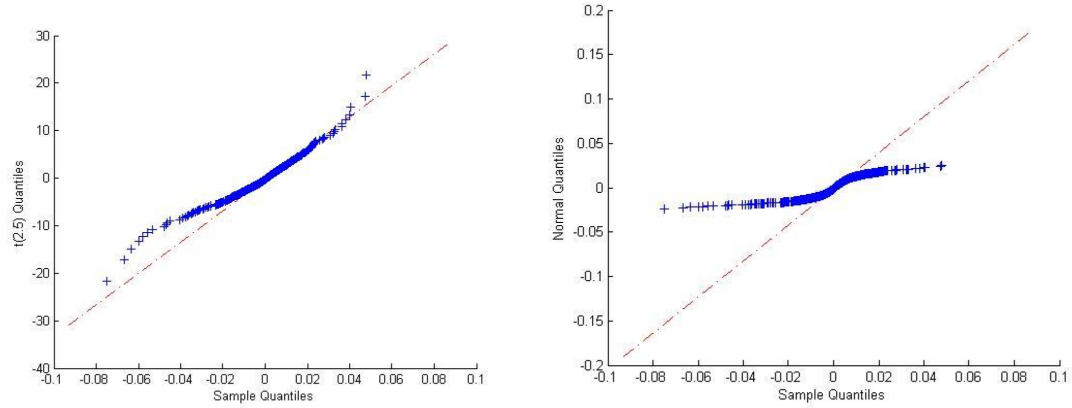


Figure 4.1.7: JPMorgan Em. Mark. Bond Index daily log-return quantiles vs. t(2,5) (left) and normal(0,1) (right).

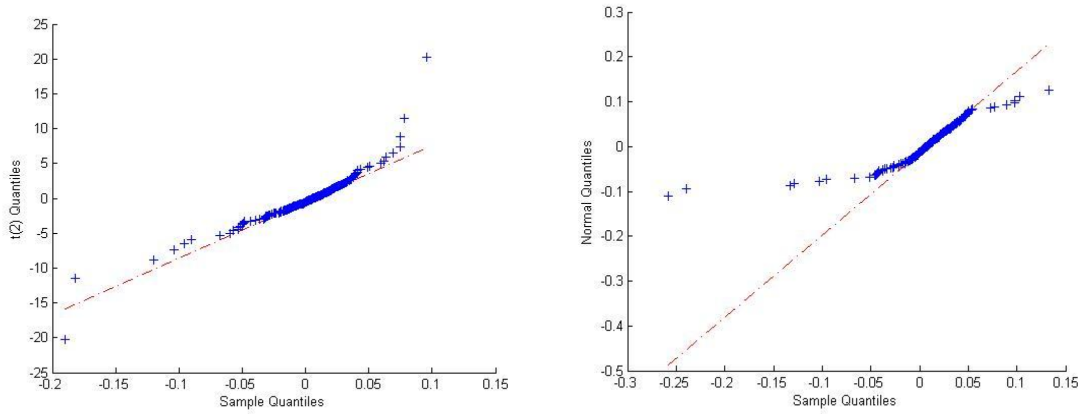


Figure 4.1.8: JPMorgan Em. Mark. Bond Index monthly log-return quantiles vs. t(2) (left) and normal(0,1) (right).

Credit Suisse High Yield Index

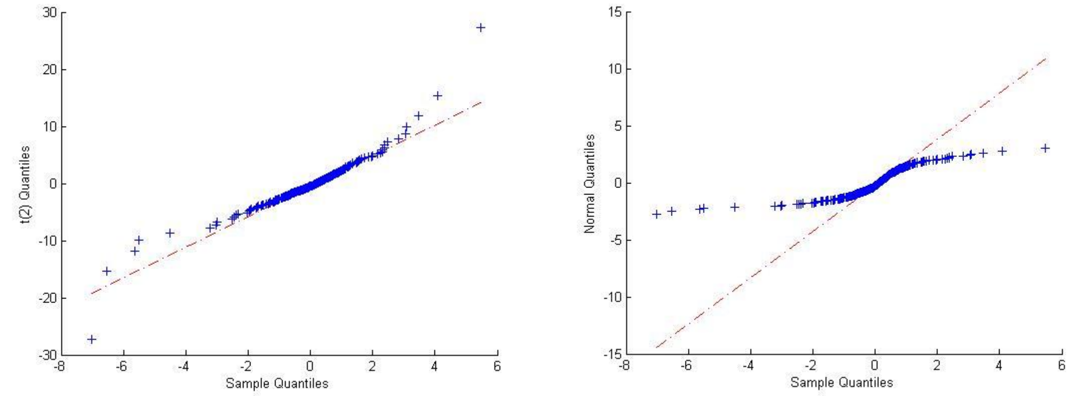


Figure 4.1.9: Credit Suisse High Yield Index weekly log-returns quantiles vs. t(2) (left) and normal(0,1) (right).

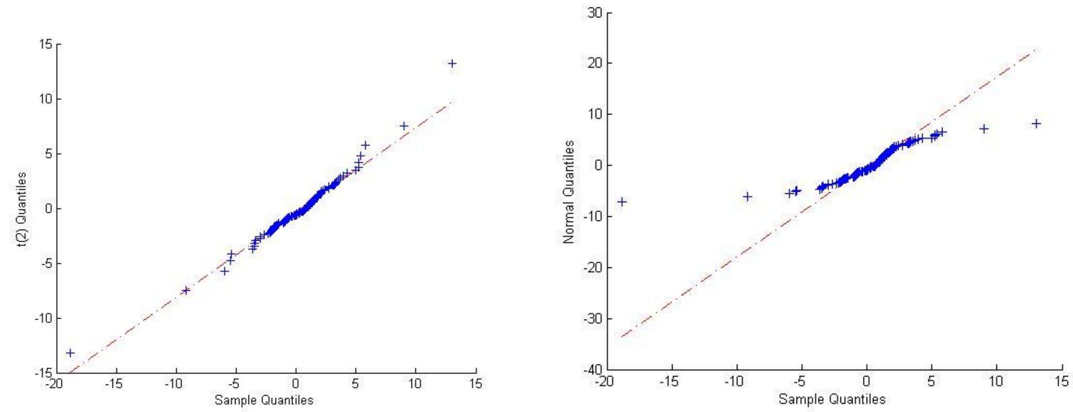


Figure 4.1.10: Credit Suisse High Yield Index monthly log-return quantiles vs. t(2) (left) and normal(0,1) (right).

Clearly, 1-day log-returns are not normal. Depending on the asset class, they tend to have a decent fit with the student-t distribution with 2-8 degrees of freedom. It is a well known fact that financial return-data have fat tails, something which we here confirm also for fixed income assets. As one would suspect more risky assets such as high-yield bonds (Credit Suisse High Yield Index) and emerging market bonds (JPMorgan Em. Mark. Bond Index) tend to be very fat-tailed with around 2 degrees of freedom whereas Swedish government bonds (OMRX-Bond Index) and European investment-grade corporate bonds (FTSE Euro Corp. Bond Index) tend to be closer to a normal distribution with 6-8 degrees of freedom. American corporate bonds (Dow Jones Corp. Bond Index) however are significantly more fat-tailed than European with approximately 4 degrees of freedom, something which is most likely explained by the significant differences in credit markets between USA and the Euro-area. European companies are more reliant on bank loans for financing, with only the largest companies issuing bonds, whereas American companies to a much larger extent use bonds for financing (Hartmann et. al., 2003).

When looking at the monthly returns the less risky assets, OMRX-Bond Index and FTSE Euro Corp. Bond Index, has reasonable fit with the normal distribution, i.e. they do not show signs of fat-tailed behavior. In the case of FTSE Euro Corp. Bond Index the data actually seem to be less fat-tailed than the normal distribution. This is however not true for the other indexes where monthly returns have approximately the same number of degrees of freedom as their respective daily returns.

4.2 Testing for Autocorrelation and Heteroskedasticity

To answer the question whether the log-returns of our data sets consists of random shocks to the asset price, i.e. the returns are a random walk, or whether there is dependence between returns, we plot the sample autocorrelation function for daily and monthly returns. To investigate heteroskedasticity, i.e. volatility clustering, we also plot the sample autocorrelation function for squared daily and monthly log-returns.

We compute the sample autocorrelation function in the form proposed by Box, Jenkins and Reinsel:
The autocorrelation of lag k is denoted r_k ,

$$r_k = \frac{c_k}{c_0}$$

$$c_k = \frac{1}{N}\sum_{t=1}^{N-k}(x_t - \mu)(x_{t+k} - \mu) \qquad k = 0,1,2 \dots K$$

Where x_t *denotes log-returns (squared in the case of heteroskedasticity plot),* μ *the mean, K the number of lags to incorporate and N the length of the vector of returns.*

We chose to use a K (number of lags to be plotted) of 250 (1 year) in the case of daily log-returns and N-1 in the case of monthly log-returns. Using longer lags is possible for daily returns, but would be of limited viability considering that what happens one specific day would with all certainty have very limited impact on the return of a day more than 250 days in the future.

To see if there is significant autocorrelation or heteroskedasticity an approximate 95% confidence interval consisting of +/- 2 standard deviation is plotted in the ACF-graphs. More than 5% of points outside this interval hence imply significant autocorrelation or heteroskedasticity.

OMRX-Bond Index

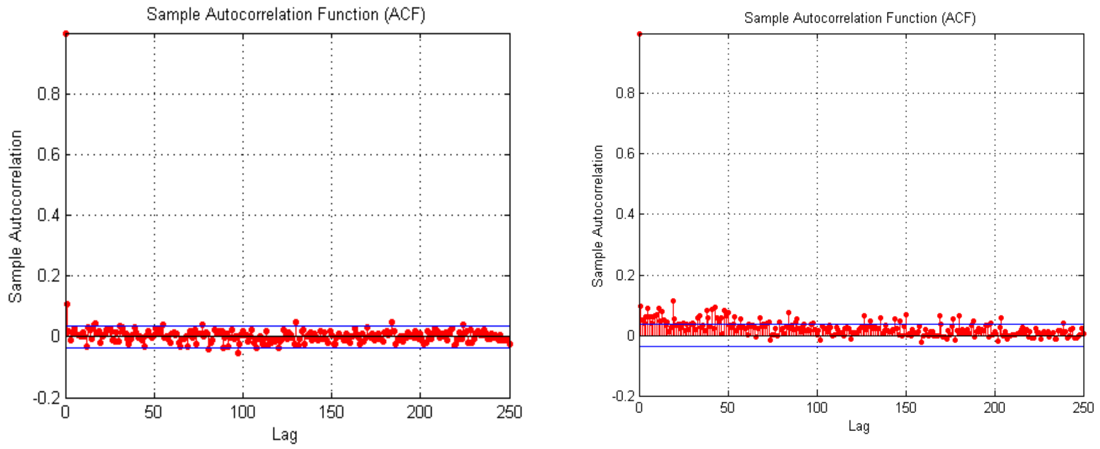


Figure 4.2.1: Sample autocorrelation function for OMRX-Bond Index daily log-returns with approximate 95% confidence interval. To the left, ACF for daily returns. To the right, ACF for squared daily returns.

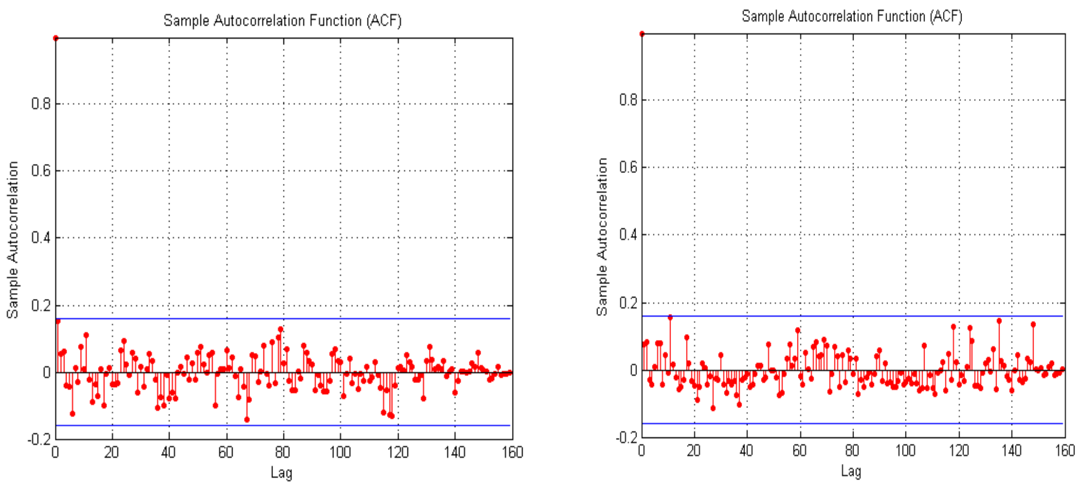


Figure 4.2.2: Sample autocorrelation function for OMRX-Bond Index monthly log-returns with approximate 95% confidence interval. To the left, ACF for monthly returns. To the right, ACF for squared monthly returns.

FTSE Euro Corp. Bond Index

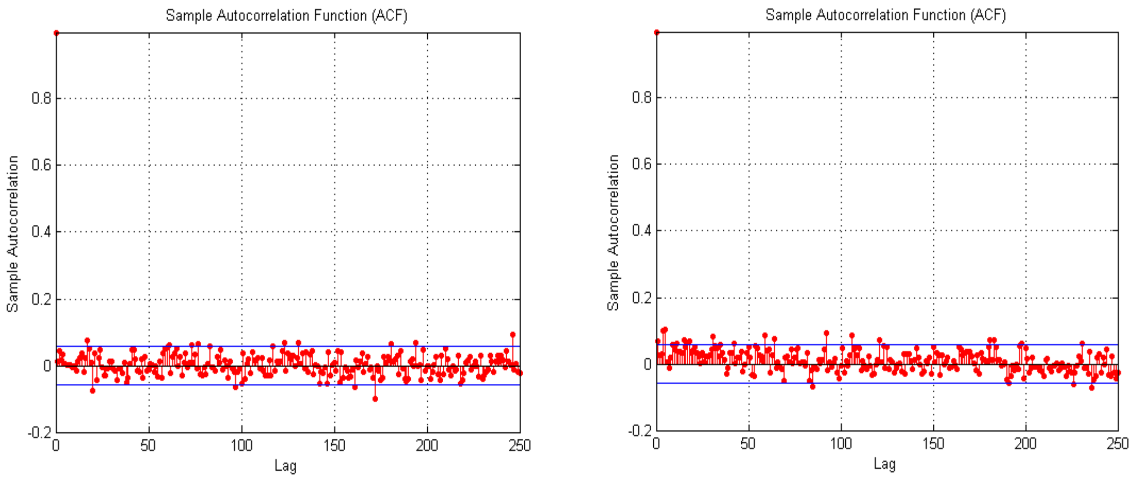


Figure 4.2.3: Sample autocorrelation function for FTSE Euro Corp. Bond Index daily log-returns with approximate 95% confidence interval. To the left, ACF for daily returns. To the right, ACF for squared daily returns.

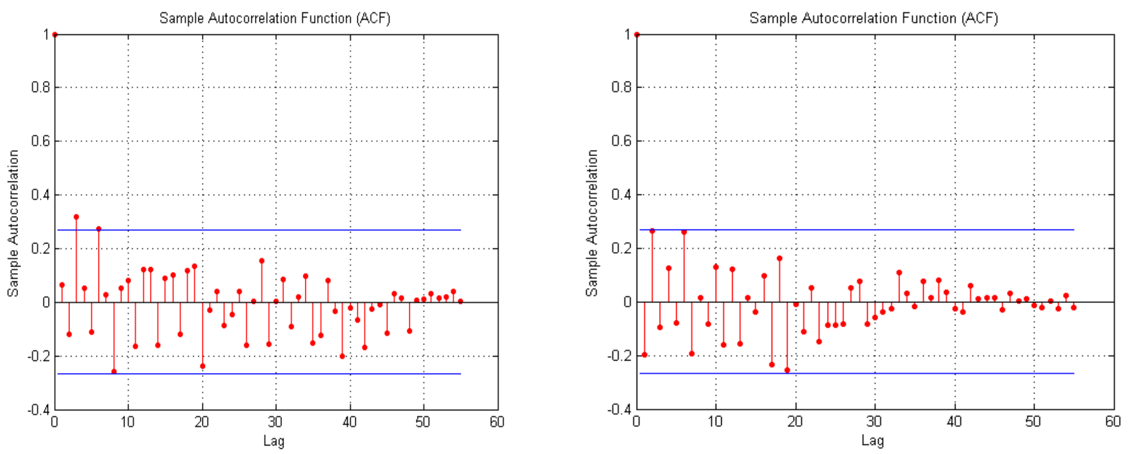


Figure 4.2.4: Sample autocorrelation function for FTSE Euro Corp. Bond Index monthly log-returns with approximate 95% confidence interval. To the left, ACF for monthly returns. To the right, ACF for squared monthly returns.

Dow Jones Corp. Bond Index

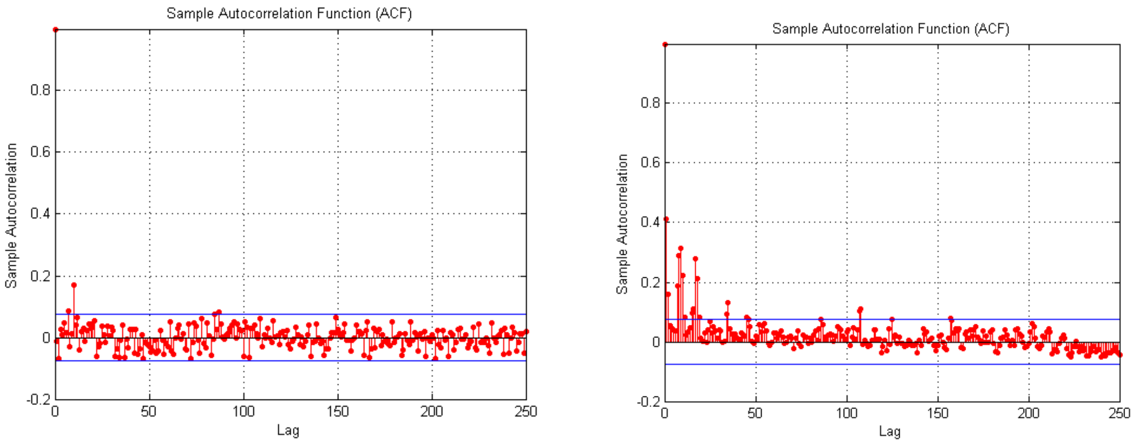


Figure 4.2.5: Sample autocorrelation function for Dow Jones Corp. Bond Index daily log-returns with approximate 95% confidence interval. To the left, ACF for daily returns. To the right, ACF for squared daily returns.

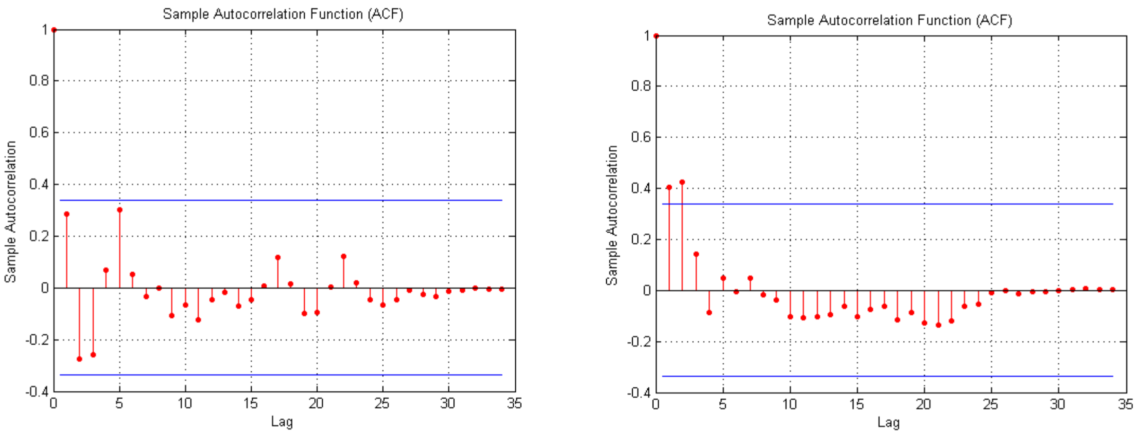


Figure 4.2.6: Sample autocorrelation function for Dow Jones Corp. Bond Index monthly log-returns with approximate 95% confidence interval. To the left, ACF for monthly returns. To the right, ACF for squared monthly returns.

JPMorgan Em. Mark. Bond Index

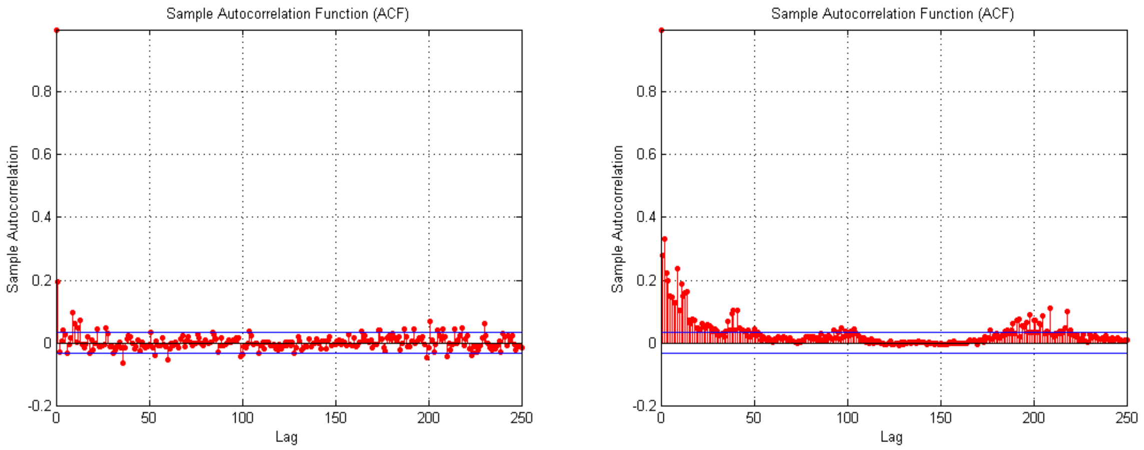


Figure 4.2.7: Sample autocorrelation function for JPMorgan Em. Mark. Bond Index daily log-returns with approximate 95% confidence interval. To the left, ACF for daily returns. To the right, ACF for squared daily returns.

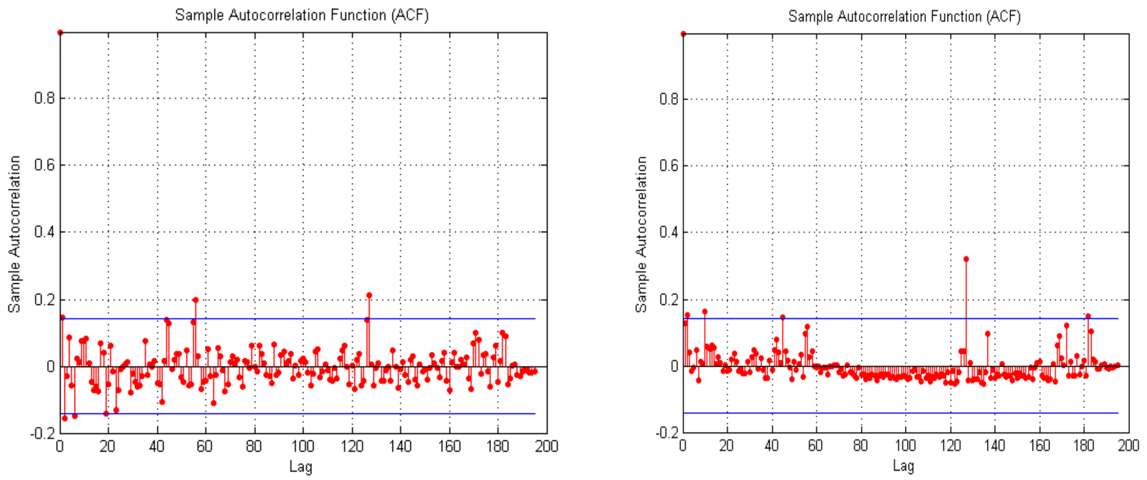


Figure 4.2.8: Sample autocorrelation function for JPMorgan Em. Mark. Bond Index monthly log-returns with approximate 95% confidence interval. To the left, ACF for monthly returns. To the right, ACF for squared monthly returns.

Credit Suisse High Yield Index

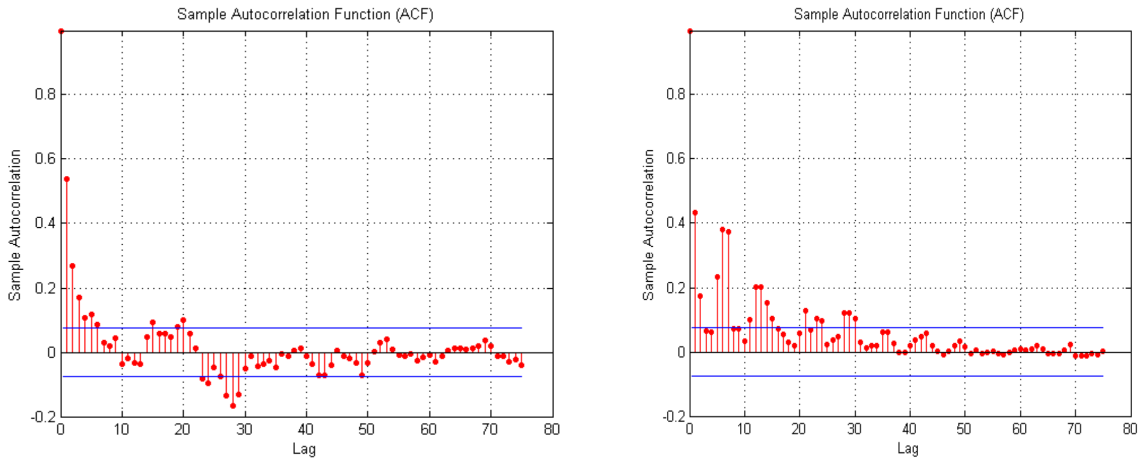


Figure 4.2.9: Sample autocorrelation function for Credit Suisse High Yield Index weekly log-returns with approximate 95% confidence interval. To the left, ACF for weekly returns. To the right, ACF for squared weekly returns.

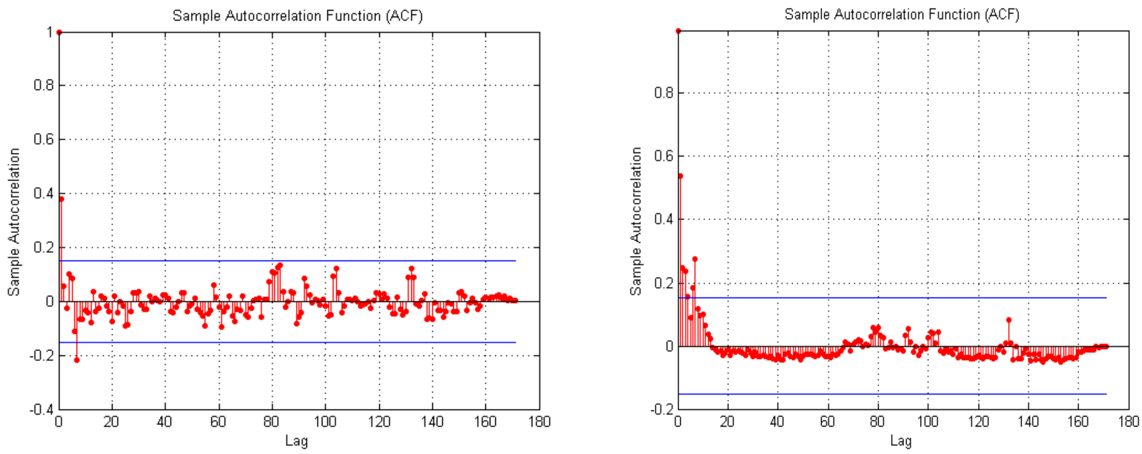


Figure 4.2.10: Sample autocorrelation function for Credit Suisse High Yield Index monthly log-returns with approximate 95% confidence interval. To the left, ACF for monthly returns. To the right, ACF for squared monthly returns.

Clearly, there is significant heteroskedasticity in the daily log-returns of our data sets. All data show signs of this, perhaps most pronounced in JPMorgan Em. Mark. Bond Index. The daily returns of JPMorgan Em. Mark. Bond Index and Credit Suisse High Yield Index also show signs of autocorrelation. Assuming independent daily returns of fixed income data hence seem inappropriate. Looking at monthly returns however, our graphs suggest that there is little evidence of any autocorrelation or heteroskedasticity, with the exception of some heteroskedasticity in the monthly log-returns of Credit Suisse High Yield Index. Assuming independence, as in a random walk, might therefore be appropriate when modeling monthly returns of fixed income assets. The existence of dependence in monthly returns cannot however be excluded, and we will in this study therefore include modeling monthly returns with time series that have autocorrelation or heteroskedasticity.

4.3 Estimating the Parameters of an AR(1)-process

We estimated the parameters of an AR(1)-process for the daily log-returns of the indices described above. When estimating the parameters of the AR-process we start by subtracting the mean from the original data, hence ending up with a zero mean process.

The parameter estimation is straight forward giving us a φ (see section 2.3) ranging from 0.01 for Dow Jones Corp. Bond Index to 0.30 for JPMorgan Em. Mark. Bond Index. All data sets had statistically significant parameters.

4.4 Estimating the Parameters of Garch(1,1) and AR(1)-Garch(1,1) Processes

The GARCH and AR-GARCH estimation was made with the same zero mean processes used above in the AR estimation. The two most interesting parameters is of course the GARCH and the ARCH parameter (we limited ourselves to estimation of GARCH(1,1) and AR(1)-GARCH(1,1) processes) who ranged from 0.70 (Credit Suisse High Yield Index) to 0.96 (OMRX-Bond Index) for the GARCH parameter and for the ARCH parameter we see range of 0.01 (OMRX-Bond Index) to 0.25 (JPMorgan Em. Mark. Bond Index). All data sets had statistically significant parameters.

4.5 Summary and Discussion of Distribution Analysis

It is a well known fact that financial return data have a tendency for fat tails and volatility clustering (i.e. heteroskedasticity). What we have shown in this chapter is that this is largely true also for daily log-returns of fixed income data. Monthly returns however are, in the case of the less risky OMRX-Bond Index and FTSE Euro Corp. Bond Index, more normally distributed, although not so in the other indices in our study. We have also shown that monthly returns are reasonably independent for all indices, with low autocorrelation and heteroskedasticity.

The appropriate method of modeling fixed income data hence depends on the time horizon of returns as well as the asset class under study. AR, GARCH and AR-GARCH methods with fat-tailed innovations might well be appropriate for daily returns (statistically significant parameters for all data sets), whereas random walk models with less fat-tailed innovations might be more appropriate for the monthly equivalent.

5. Scaling of 1-day to 10-day VaR with Simulated Data

This chapter deals with the scaling of 1-day Value-at-Risk to 10-day Value-at-Risk. We compare performance of the square-root-of-time rule to a number of empirical methods suggested by Roger Kaufmann (Kaufmann, 2004) described below in scaling 1-day VaR of simulated data from the distributions fitted in chapter 4. One might argue that using an EVT (Extreme Value Theory) scaling method would be appropriate in this comparison, but the implementation of these methods demands input from the user in the form of e.g. somewhat subjective graphical procedures, making the automated process for a good comparison methodology difficult. The scaled VaR's are then compared to a "true" 10-day VaR, derived either analytically (when possible) or from a very large number of simulations. The distributions used to generate simulated data are random walks with standard normal and student-t innovations as well as the AR-, GARCH- and AR-GARCH processes. The parameters for the distributions are derived from chapter 4, in order to test the procedures on data that are similar to real fixed income data. This we believe is crucial for obtaining results that have validity in real-world applications.

The scaling methods:

- *Bootstrap*

  Randomly and with repetition 10 values are chosen $(X_{i_1}, \dots, X_{i_{10}})$ out of the sample of 500 values and these 10 values are then summed up: $Y_n := \sum_{k=1}^{10} X_{i_k}$. We repeat this procedure 10,000 times and then evaluate the empirical 99% quantile of $(Y_n)_{n=1,\dots,10000}$.
  An underlying assumption of Bootstrap is independent data points, and hence it should perform well for random walk-like data sets.

- *Independent Resampling*

  Here we choose 10 weakly dependent daily log-returns: choose $X_{i_1}, \dots, X_{i_{10}}$ such that $min_{j<k}(i_k - i_j) \geq 10$. These 10 values are then summed up: $Y_n := \sum_{k=1}^{10} X_{i_k}$. We repeat this procedure 10,000 times and then the empirical 99% quantile is evaluated.
  The procedure can be seen as a counter method to dependent resampling below.

- *Dependent Resampling*

  In this method 10 strongly dependent values are chosen out of the sample of 500 values. For n=1,…, 481, we pick $X_{i_1}^n, \dots, X_{i_{10}}^n$, where $i_k \in [n, n+19]$. These then values are summed up: $Y_n := \sum_{k=1}^{10} X_{i_k}^n$. We aim have ca 10,000 values to work with and therefore we repeat the procedure 22 times. Then the empirical 99% quantile is evaluated.
  The logic with this procedure is to preserve the dependency structure inherent in the data set. Should in theory be good for data with high degree of dependency.

- *Non-overlapping periods*

  We have 50 non-overlapping periods 10-day log-returns:
  $Y_n = \sum_{i=0}^{9} X_{10n-i}$ $(n = 1,2,\dots,50)$. Using these we may then evaluate the 99% empirical quantile.
  The theoretically best method given infinite data, however problematic due to lack of 10-day return data in real world application.

- *Overlapping periods*

  We have 491 overlapping 10-day log-returns: $Y_n \coloneqq \sum_{i=0}^{9} X_{n+i}$ $(n = 1,2,\dots,491)$. Using these we may then evaluate the empirical 99% quantile.
  Simply used as a complement to non-overlapping periods to get more data points. Does not take into account the dependence of the overlapping periods.

- *Square-root of time*

  Using our sample of daily log-returns we evaluate the empirical quantile and this daily VaR is then scaled by the factor $\sqrt{10}$ to get the 10-day VaR.
  The analytical solution of 10-day VaR for random walk with normal innovations data.

Since the purpose of this chapter is to give a theoretical view on the scaling methods and their respective performance for different distributions rather than to come up with actual scaled VaR-figures, we limited the study to standard normal, student-t with 3 and 6 degrees of freedom and AR (φ=0.1), GARCH (a=0.1, b=0.83) and AR-GARCH (φ=0.1, a=0.1, b=0.83) parameters which are reasonably applicable to all the data sets in chapter 3. These distributions we feel give us a good collection of typical fixed income log-return distributions to test the different scaling methods on.

In deciding on which amount of data to use when estimating the 1-day VaR one can either take a more realistic approach and simulate a limited number of daily returns (Kaufmann, 2004) or simulate a very large amount of data to get a more correct 1-day VaR-measure to scale (Kristofersen et. al.,1998). Since the purpose of this study is to provide a framework for real-world scaling applications, where data is limited, we choose to test the scaling methods on 500 (2 years) simulated log-returns. This we feel is a reasonable number considering statistical significance for the 1-day VaR as well as mimicking real world data availability.

The procedure devised by Roger Kaufmann in detail:

1. Simulation of 10^6 daily log-returns from the different distributions chosen.
2. The true 10-day VaR is evaluated from this series (given analytically in the case of standard-normal random walk).
3. 10-day VaR on 500 simulated 1-day returns is calculated 1000 times with all the different scaling procedures.
4. The results of the different procedures are compared to the true VaR as well as the square-root-of-time method with a graphical procedure as well as the mean.

The results of the simulations are summarized in figures 5.1.1-9 below. The plots show the outcome of the VaR-scaling methods for every simulation of 500 daily log-returns, with the square-root-of-time rule on the x-axis and the respective comparison method on the y-axis. The vertical and horizontal lines represent the true 99% 10-day VaR for the respective distributions. The dotted line goes through ($VaR_{99\%}, VaR_{99\%}$) and has slope 1. The slope of the solid line is determined by the ratio of the absolute value of the sum of deviances from the true VaR for the comparison method to the sum of deviances for the square-root-of-time rule,

$$S = \frac{\sum_{i=1}^{1000} \left| y^i - VaR_{99\%} \right|}{\sum_{i=1}^{1000} \left| x^i - VaR_{99\%} \right|}$$

A slope S larger than 1 implies that the square-root-of-time is the superior method for the distribution at hand. Regarding the clustering of the observations, a majority of points to the right (left) of the true VaR means that square-root-of-time overestimates (underestimates) the VaR, whereas a majority of points above (below) the true VaR means that the comparison method is overestimating (underestimating) the true VaR.

5.1 Scaling of Random Walk with Normal Innovations Data

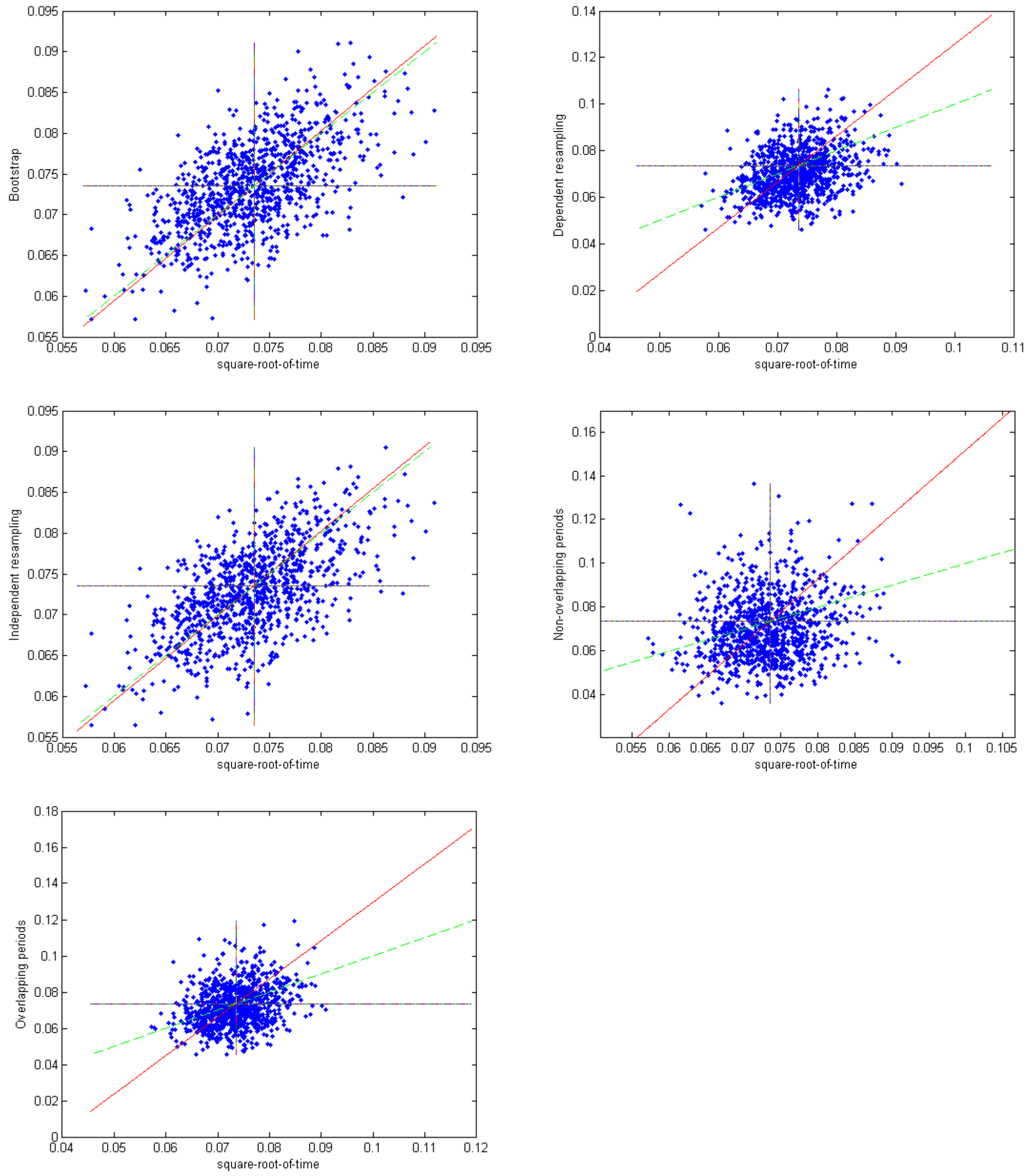


Figure 5.1.1: Random Walk Normal(0,1) innovations

As expected, the slope of the full-drawn line, indicating the performance of the respective scaling-methods relative to the square-root-of-time procedure, is greater than 1 (line is above dotted line) for all methods. This result is in line with theory as the square-root-of-time rule is the analytical solution to scaling the VaR of a normal random-walk time series. Independent resampling as well as bootstrap gives a very good estimation of the true 10-day VaR, with slopes very close to 1.

Random Walk Normal Innovations		
Method	Mean	STD
Bootstrap	0,073529	0,005388
Dependent resampling	0,070515	0,009659
Independent resampling	0,072828	0,005346
Nonoverlapping periods	0,070447	0,014789
Overlapping periods	0,071748	0,010773
Square-root-of-time	0,073262	0,005198
True (analytical)	0,0736	-

Table 5.1.1, mean and standard deviations of scaled 99% 10-day VaR for random walk with normal innovations.

Looking at the mean of 99% 10-day VaR using the different procedures one can see that, as expected, the square-root-of-time produces an estimate that is close to the true value in the case of random walk-data with normal innovations. The bootstrap method also performs well in this aspect, while the other methods tend to underestimate VaR somewhat. The standard deviation is in the same region for square-root-of-time, bootstrap and independent resampling, while the other methods have about twice that value, indicating that they are more unreliable as 10-day VaR estimating procedures.

5.2 Scaling of Random Walk with Student-t Data

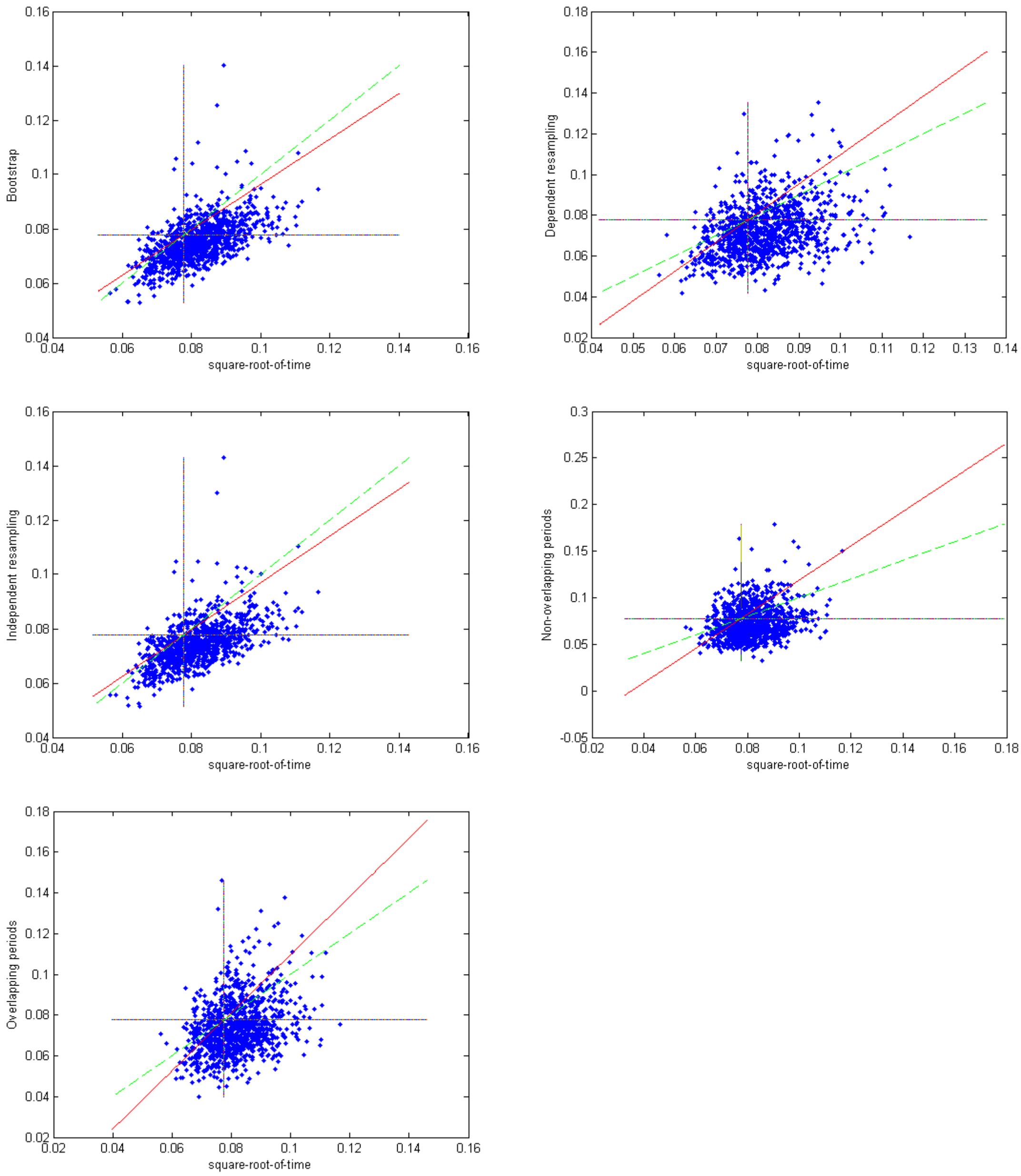


Figure 5.1.2: Random walk Student-t(6) innovations

From figure 5.1.2 above we see the satisfactory performance of independent resampling and bootstrap as in the case of normal data. The square-root-of-time rule somewhat overestimates the VaR, and is here outperformed by bootstrap and independent resampling (full-drawn line slope less than 1). The other methods perform poorly on student-t data.

Random Walk Student-t(6) Innovations		
Method	Mean	STD
Bootstrap	0.075145	0.007963
Dependent resampling	0.072605	0.012656
Independent resampling	0.074368	0.007895
Nonoverlapping periods	0.073659	0.017376
Overlapping periods	0.073976	0.013256
Square-root-of-time	0.081602	0.00898
True (simulated)	0.0754	-

Table 5.1.2, Mean and standard deviations of scaled 99% 10-day VaR for random walk with t(6) innovations.

The results from figure 5.1.2 above are confirmed when looking at the mean of the 99% 10-day VaR estimates for random walk-data with t(6) innovations. The bootstrap and independent resampling methods have means close to the true value, while the square-root-of-time and the other methods overestimate respective underestimate the VaR. Looking at the standard deviations of the different methods one can conclude that, as in the case with normal innovations, bootstrap, independent resampling and square-root-of-time are the most stable estimating procedures.

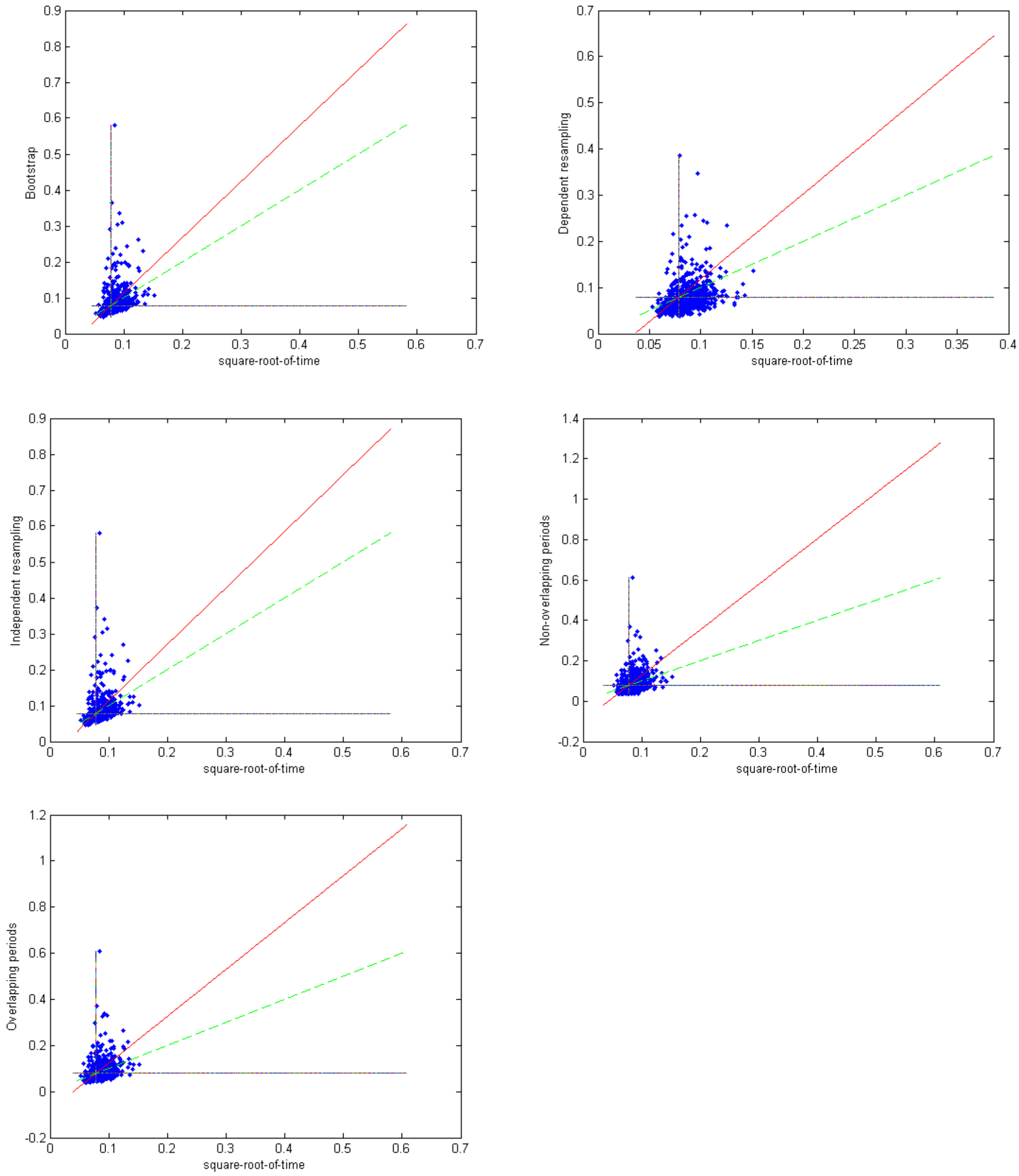


Figure 5.1.3, Random walk Student-t(3) innovations

When looking at the graphs in figure 5.1.3 with student-t data with 3 degrees of freedom, none of the methods seem to give a better estimate of the 10-day VaR than the square-root-of-time rule, all full-drawn line slopes being significantly greater than one. Of the other methods Bootstrap and independent resampling seem to give the best results as with previous data sets.

Random Walk Student-t(3) Innovations		
Method	Mean	STD
Bootstrap	0.081586	0.034237
Dependent resampling	0.074927	0.030168
Independent resampling	0.08097	0.034427
Nonoverlapping periods	0.081258	0.040027
Overlapping periods	0.081969	0.038398
Square-root-of-time	0.084272	0.013454
True (simulated)	0.0776	

Table 5.1.3, mean and standard deviations of scaled 99% 10-day VaR for random walk with t(3) innovations.

Looking at the means of the different estimation procedures, it is noticeable that all methods except for dependent resampling overestimate the 99% VaR for random walk-data with t(3) innovations. Independent resampling and dependent resampling comes closest to the true value. Noticeable is also the high standard deviation of all methods compared to when using t(6) innovations, with the square-root-of-time method being the most stable.

5.3 Scaling of AR(1) Data

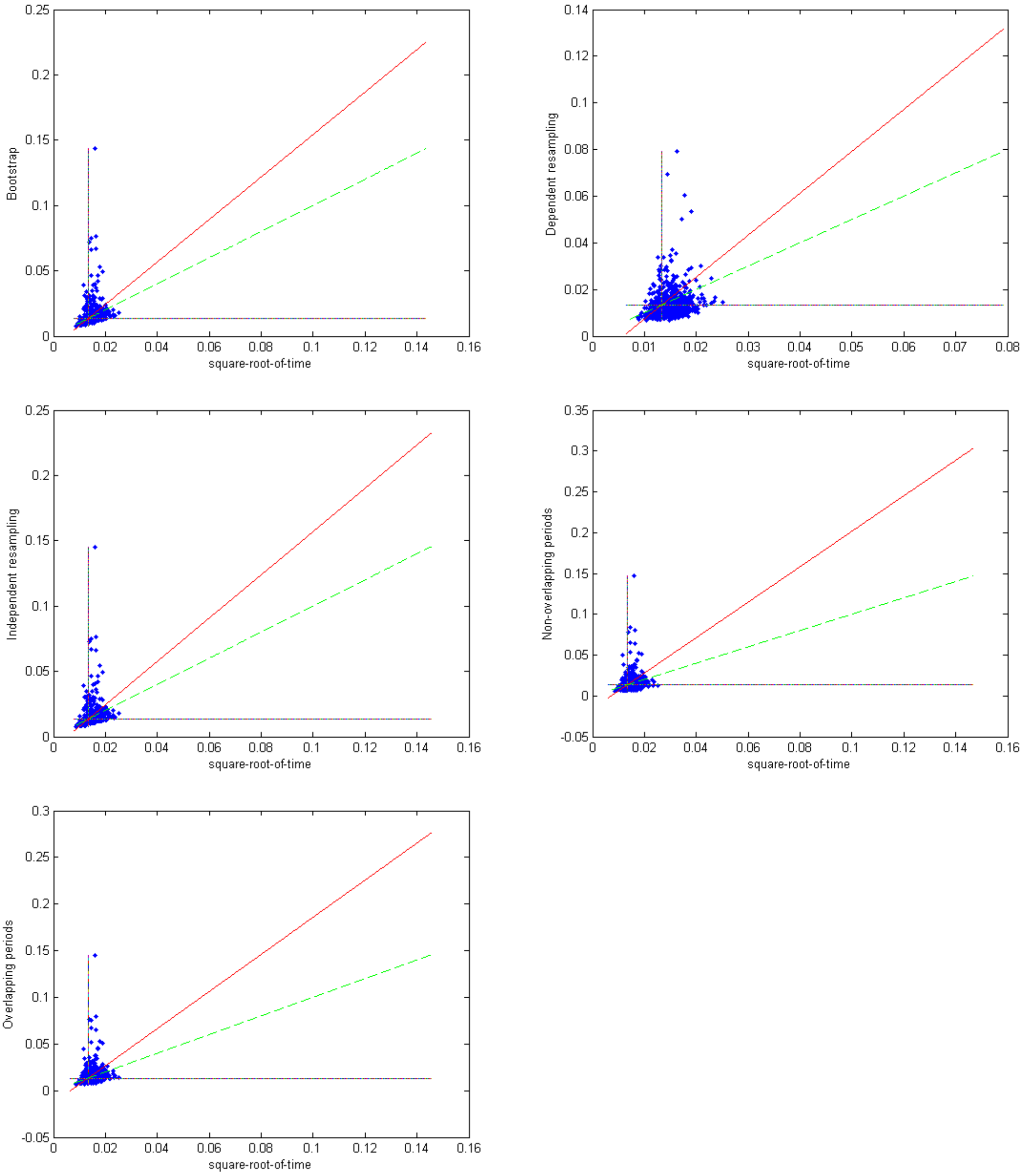


Figure 5.1.4: AR(1) process, φ=0.1, student-t(3) innovations

AR(1)-data with student-t(3) innovations give the same results as with the student-t(3) random walk, none of the methods outperform the square-root-of-time rule, bootstrap and independent resampling performing slightly better than the others (slope of full-drawn line closer to 1).

AR(1) Student-t(3) Innovations		
Method	Mean	STD
Bootstrap	0.014218	0.007415
Dependent resampling	0.01294	0.005522
Independent resampling	0.014103	0.007502
Nonoverlapping periods	0.014077	0.008171
Overlapping periods	0.014112	0.007853
Square-root-of-time	0.014462	0.002284
True (simulated)	0.0128	-

Table 5.1.4, mean and standard deviations of scaled 99% 10-day VaR for an AR(1)-process with t(3) innovations.

Comparing the mean of the different scaling-procedures for 99% VaR with AR(1)-data with t(3) innovations, we see that dependent resampling comes closest to the true VaR, the other methods overestimating somewhat. This is possibly a consequence of the dependent resampling capturing the inherent autocorrelation of the AR(1)-process. Square-root-of-time is as with previous data the most stable (i.e. lowest standard deviation) method of scaling 10-day VaR.

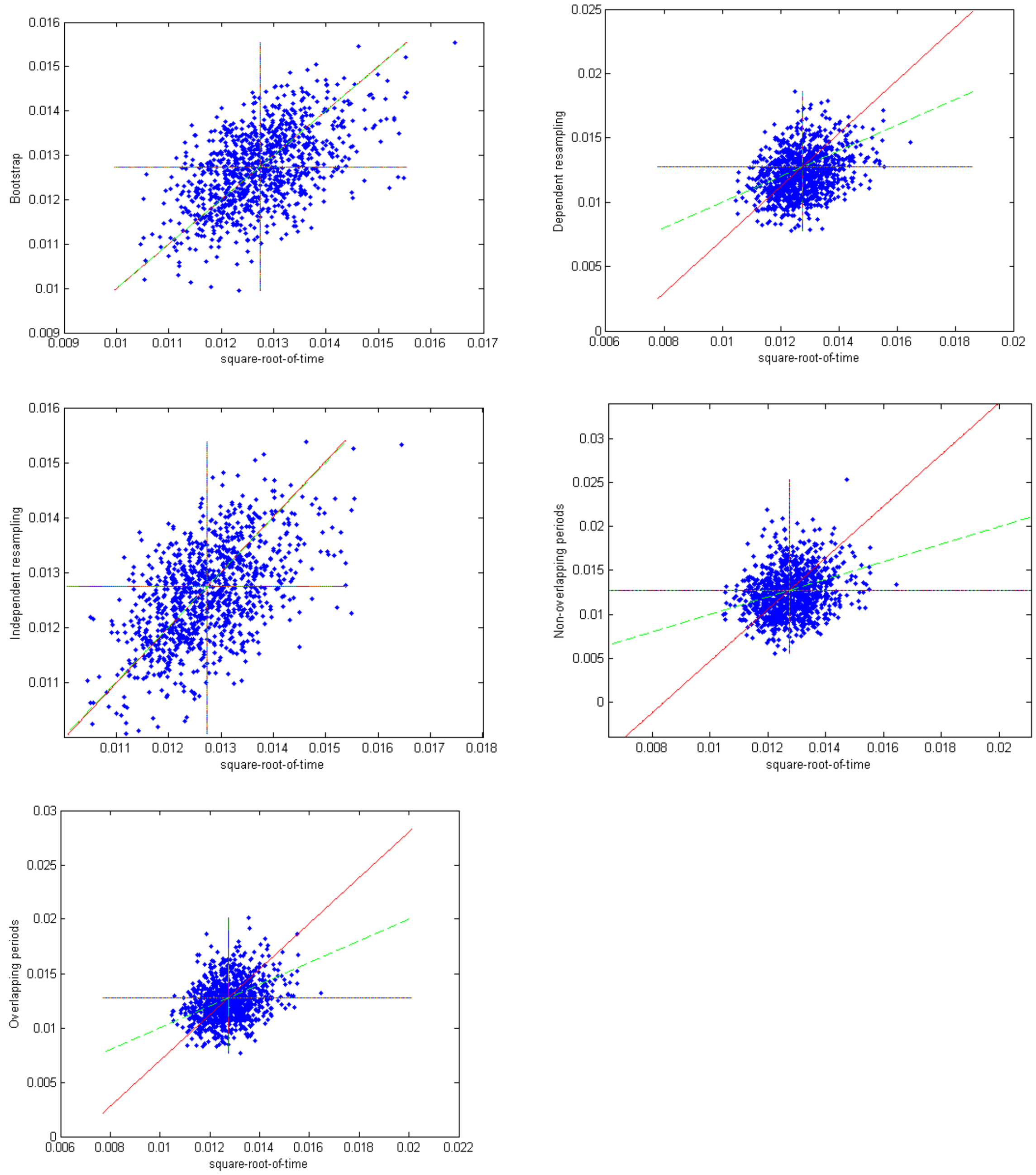


Figure 5.1.5: AR(1) process, φ=0.1, normal innovations

Given AR(1) data with normal innovations, independent resampling and bootstrap seem to be equivalent to the square-root-of time rule (slope of full-drawn line equal to 1). The other methods perform poorly.

AR(1) Normal Innovations		
Method	Mean	STD
Bootstrap	0.012716	0.000873
Dependent resampling	0.012248	0.001719
Independent resampling	0.0126	0.000869
Nonoverlapping periods	0.012444	0.002567
Overlapping periods	0.012497	0.001826
Square-root-of-time	0.012748	0.000878
True (simulated)	0.0127	-

Table 5.1.5, mean and standard deviations of scaled 99% 10-day VaR for an AR(1)-process with normal innovations.

Bootstrap, independent resampling and square-root-of-time all come very close to the true VaR when comparing the means, which is congruent with the results in figure 5.1.5 above. Noticeable is the weak performance of dependent resampling as compared to when scaling AR(1)-data with t(3) innovations. Comparing standard deviations of the methods bootstrap, independent resampling and square-root-of-time are the most stable.

5.4 Scaling of GARCH(1,1) Data

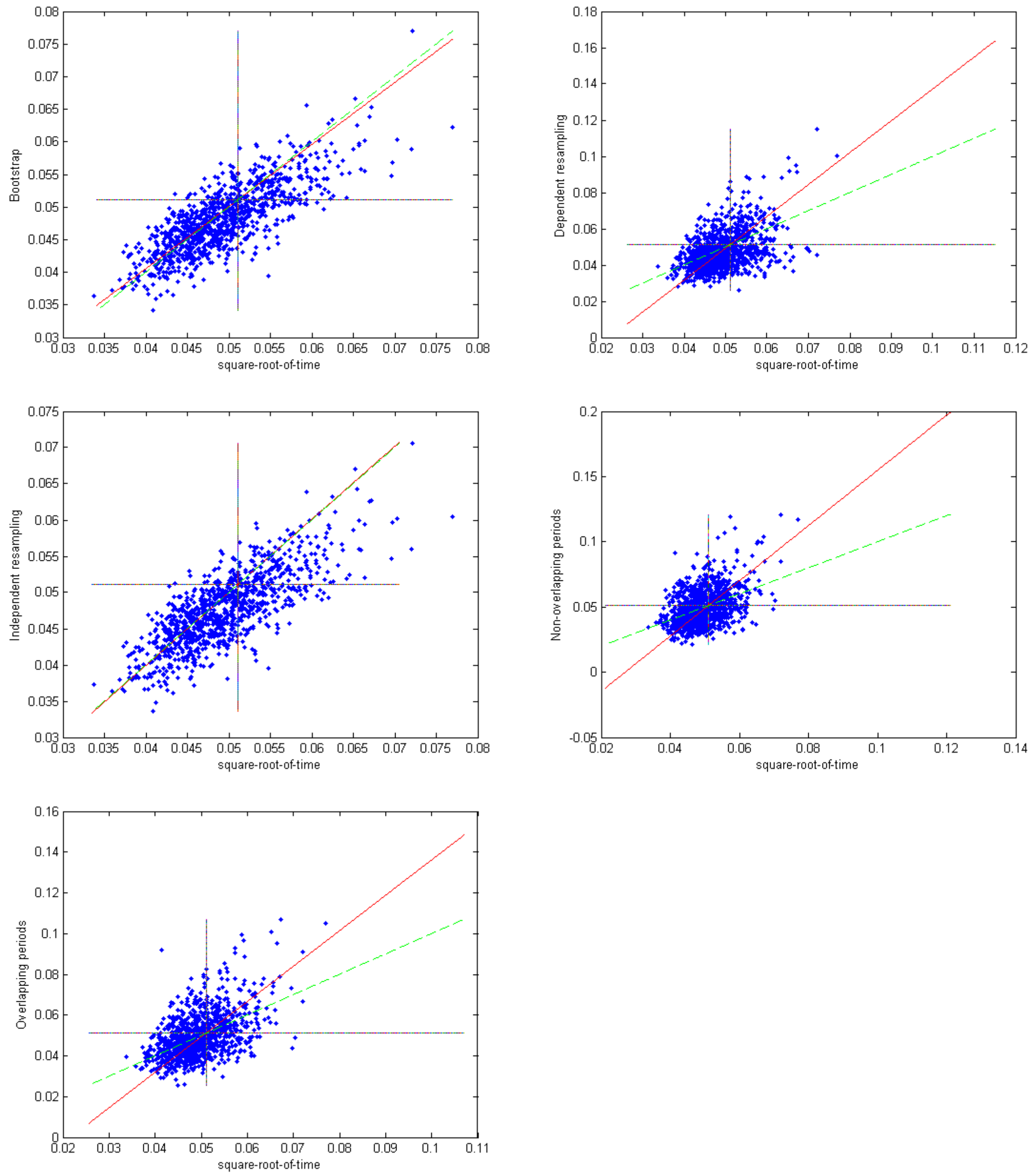


Figure 5.1.6: GARCH(1,1) process, a=0.1, b=0.83, normal innovations

For GARCH(1,1) data with normal innovations bootstrap performs slightly better than the square-root-of-time rule, while independent resampling seem to be equivalent to the latter.

GARCH(1,1) Normal Innovations		
Method	Mean	STD
Bootstrap	0.048184	0.005023
Dependent resampling	0.047792	0.010461
Independent resampling	0.047531	0.004944
Nonoverlapping periods	0.049904	0.013582
Overlapping periods	0.049724	0.011085
Square-root-of-time	0.049373	0.00591
True (simulated)	0.0511	-

Table 5.1.6, mean and standard deviations of scaled 99% 10-day VaR for a GARCH(1,1)-process with normal innovations.

Given the means of the different estimating procedures all methods seem to underestimate the 10-day VaR for GARCH(1,1)-data with normal innovations. Noticeable is that bootstrap differs more from the true VaR than the square-root-of-time method when comparing means although it outperformed square-root-of-time in the graphs above (figure 5.1.6), where we compare sum of the absolute values of the deviation from the true VaR.

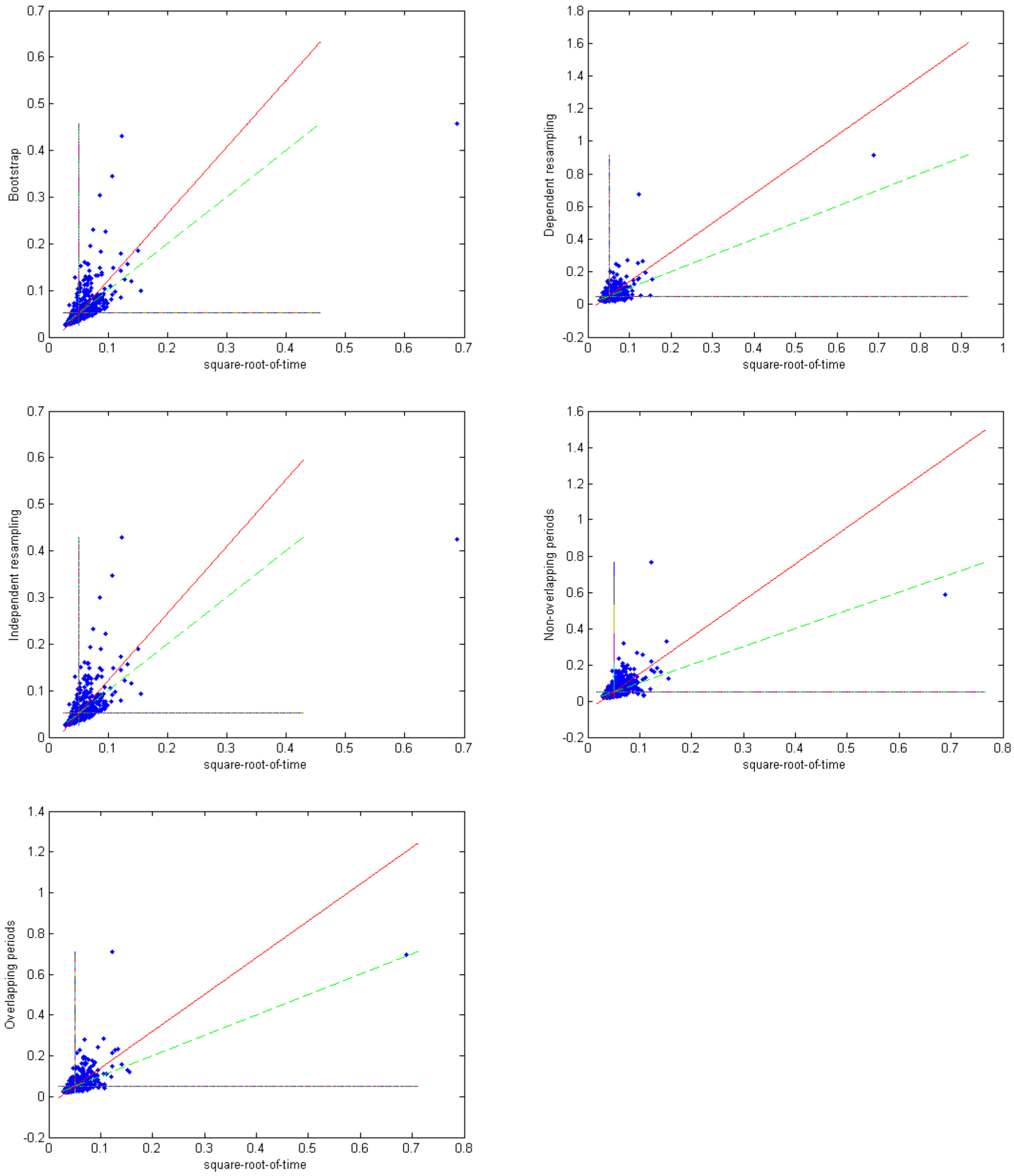


Figure 5.1.7: GARCH(1,1) process, a=0.1, b=0.83, student-t(3) innovations

As with the student-t(3) random walk and AR-process with student-t(3) innovations all methods are inferior to the square-root-of-time rule for GARCH(1,1) data with student-t(3) innovations given the graphs in figure 5.1.7.

GARCH(1,1) Student-t(3) Innovations		
Method	Mean	STD
Bootstrap	0.052138	0.031631
Dependent resampling	0.050469	0.044171
Independent resampling	0.051486	0.031205
Nonoverlapping periods	0.056412	0.044356
Overlapping periods	0.055189	0.042401
Square-root-of-time	0.054305	0.025256
True (simulated)	0.0511	-

Table 5.1.7, mean and standard deviations of scaled 99% 10-day VaR for a GARCH(1,1)-process with student-t(3) innovations.

Comparing the means of the scaling procedures, boostrap, independent resampling and dependent resampling come closest to the true VaR, while the other methods overestimate the 10-day VaR for GARCH(1,1)-data with t(3)-innovations. The square-root-of-time method however has a smaller standard deviation, possibly explaining its superior performance when comparing the sum of the absolute values of deviations from the true VaR in figure 5.1.7 above.

5.5 Scaling of AR(1)-GARCH(1,1) Data

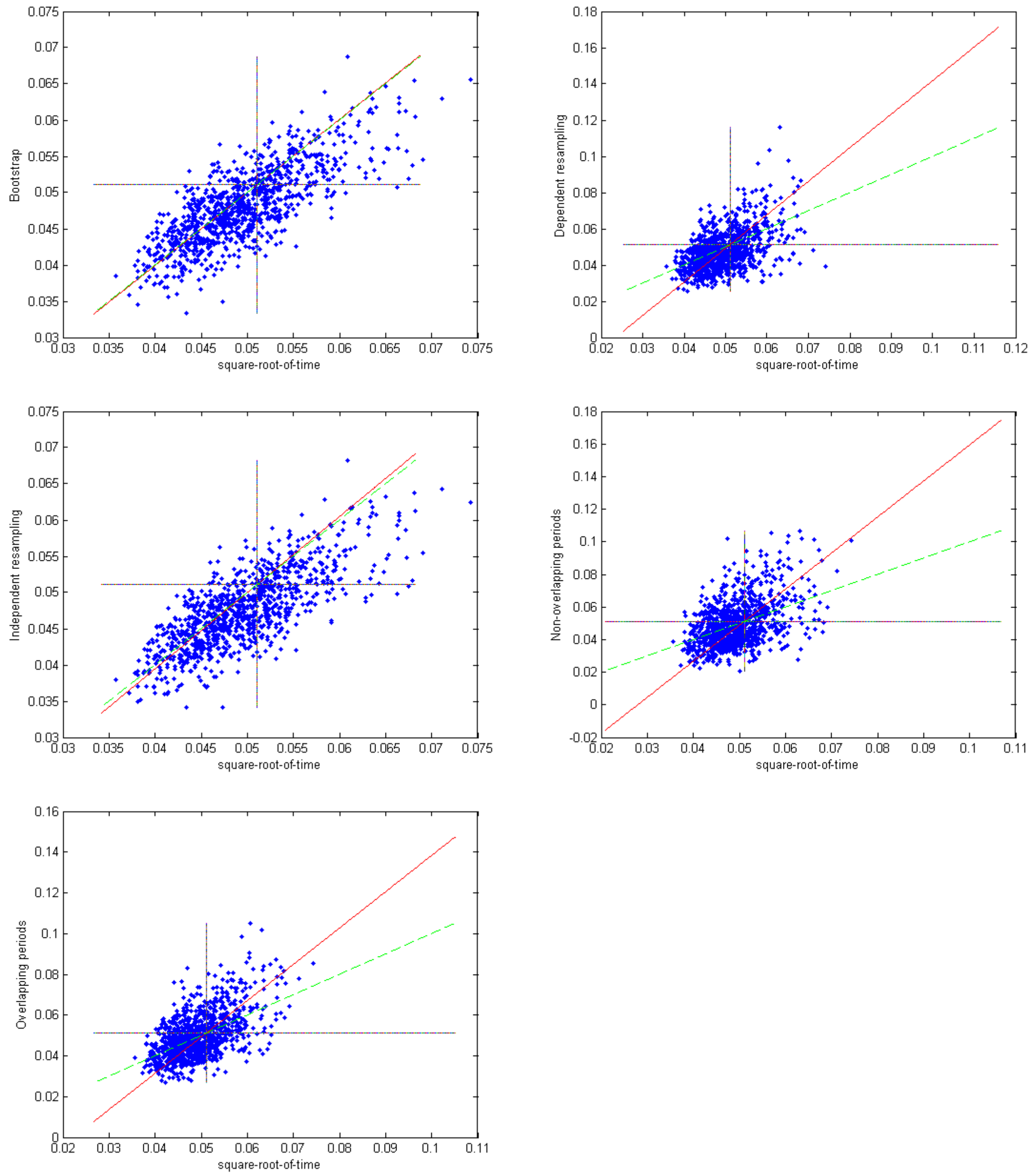


Figure 5.1.8: AR(1)-GARCH(1,1) process, φ=0.1, a=0.1, b=0.83, normal innovations

As with the other distributions with normal innovations, bootstrap and independent resampling perform in line with the square-root-of-time rule (slopes close to 1) for AR(1)-GARCH(1,1) with normal innovations.

AR(1)-GARCH(1,1) Normal Innovations		
Method	Mean	STD
Bootstrap	0.048067	0.005207
Dependent resampling	0.047691	0.010823
Independent resampling	0.047484	0.005078
Nonoverlapping periods	0.049498	0.013754
Overlapping periods	0.049844	0.011138
Square-root-of-time	0.049511	0.00589
True (simulated)	0.0511	-

Table 5.1.8: mean and standard deviations of scaled 99% 10-day VaR for an AR(1)-GARCH(1,1)-process with normal innovations.

Comparing the means all methods underestimate 10-day VaR for AR(1)-GARCH(1,1)-data with normal innovations. Given the standard deviations bootstrap, independent resampling and square-root-of-time give the most stable estimations

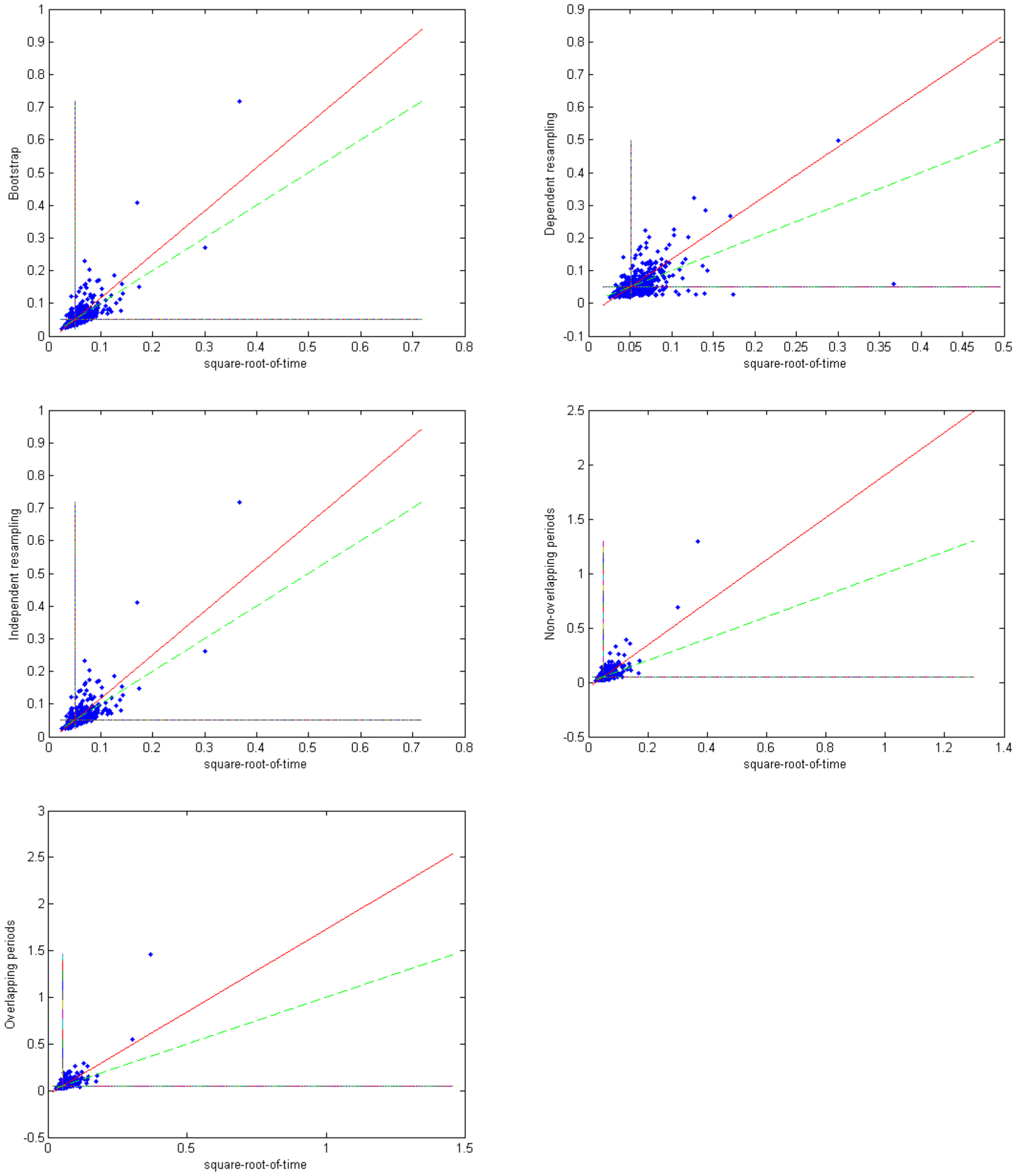


Figure 5.1.9: AR(1)-GARCH(1,1) process, φ=0.1, a=0.1, b=0.83 Student-t(3) innovations

As with the other distributions with student-t(3) innovations the square-root-of-time rule seem to outperform the other methods for AR(1)-GARCH(1,1) data.

AR(1)-GARCH(1,1) Student-t(3) Innovations		
Method	Mean	STD
Bootstrap	0.051079	0.032764
Dependent resampling	0.049101	0.032458
Independent resampling	0.050462	0.0327
Nonoverlapping periods	0.055825	0.055453
Overlapping periods	0.055477	0.055016
Square-root-of-time	0.054689	0.020394
True (simulated)	0.0511	-

Table 5.1.9: mean and standard deviations of scaled 99% 10-day VaR for an AR(1)-GARCH(1,1)-process with student-t(3) innovations.

Given the means, bootstrap comes closest to the true VaR for AR(1)-GARCH(1,1) with t(3) innovations. Square-root-of-time overestimates the 10-day VaR somewhat, its lower standard deviation possibly explaining its superior performance over bootstrap when using the graphical comparison method in figure 9 above.

5.6 Summary of Results and Conclusions

The results per scaling method

Bootstrap:
Of the so called empirical methods bootstrap is the best performing for all data sets except AR(1) with t(3) innovations. It performs well for all data sets both considering our graphical comparison method, where it outperforms square-root-of-time for random walk with t(6) innovations and GARCH(1,1) data with normal innovations, and with regards to the deviation of the mean from the true VaR as well as when considering the stability of the estimations (i.e. relatively low standard deviation).

Dependent resampling:
Dependent resampling performs poorly for most data sets, consistently underestimating the 10-day VaR. The exception is AR(1) data with t(3) innovations, where it is the best performing method. This could be a consequence of the dependent resampling capturing the autocorrelation in the AR-process. However, it does not perform as well for the AR(1) data with normal innovations.

Independent Resampling:
Performs in line with bootstrap but slightly less conservative, especially good estimates for random walk with t(3) innovations.

Non-overlapping periods:
As expected, simply using non-overlapping periods to estimate 10-day VaR gives too few data points for a reliable estimation, something which is evident in the high standard deviation of the VaR estimates (highest of all methods). Performs worse than both bootstrap and square-root-of-time for all data sets.

Overlapping periods:
Slightly better performance than non-overlapping periods with somewhat lower standard deviation.

Square-root-of-time:
Lowest standard deviation (most stable estimate) for all data, outperforms or equal all the other methods except for random walk with t(6) innovations, GARCH(1,1) with normal innovations and AR(1) with t(3) innovations, especially given the results in the from graphical procedure. The good performance relative to the other methods are most pronounced when scaling data with t(3) innovations. Good all-round method which is on the conservative side, especially relatively to the other methods when comparing mean, in a majority of the data.

A clear tendency in our analysis is for the square-root-of-time to outperform the other methods when looking at fat-tailed data (t(3) innovations) with the graphical procedure. Looking at mean, however, square-root-of-time seems to overestimate the VaR, being too conservative. The deviating results are consequences of the fact that the graphical procedure looks at the absolute value of the sum of deviations from the true VaR. Hence, even though the means of the estimations are off, the smaller variance of square-root-of-time compared to other methods makes it favorable.

To conclude, square-root-of-time performs surprisingly well in scaling the 99% VaR of our different simulated data sets and is easy to implement. Though it overestimates VaR for fat-tailed data, it is on the conservative side. These results are largely congruent with those of Kaufmanns and Embrechts studies. Combined with the Bootstrap and Independent resampling methods for GARCH(1,1)-like data with normal innovations and possibly random walk-like data with student-t innovations with more than 3 degrees of freedom , one has a good set of tools in scaling Value-at-Risk. Applying this framework to the indices described in chapter 4 and disregarding autocorrelation and heteroskedasticity, square-root-of-time should be suitable for the more fat-tailed data sets with around 2 degrees of freedom (JPMorgan Em. Mark. Bond Index, Credit Suisse High Yield Index, Dow Jones Corp. Bond Index) in obtaining 10-day VaR while bootstrap could be a good alternative for the more normally-distributed OMRX-Bond Index and FTSE Euro Corp. Bond Index indices. These methods do however not take into account dependency between returns. In chapter 4 we saw that there is significant heteroskedasticity in the daily log-returns of our data, implying that using GARCH-like methods might be a viable alternative in scaling 10-day VaR for these assets.

6. Quarterly and Yearly VaR

In this chapter we attempt to estimate quarterly and yearly VaR for the indices described in chapter 3 and 4, comparing two of the scaling methods used for 10-day VaR in chapter 5 as well as utilizing random walk and AR-processes to obtain quarterly and yearly VaR.

In scaling the VaR longer time periods we proceed with a method developed by Roger Kaufmann (Kaufmann, 2004). The appropriate way of obtaining a K-month VaR estimate is a two-step method where one first models h-day returns with 1<h<K and estimate the h-period VaR. One then proceeds with an appropriate scaling method to obtain the desired K-month VaR measure. In choosing the appropriate h-day period one has to take into consideration both the number of data points needed for a reliable estimation of h-day VaR as well as the degree of independence between periods (important for random walk scaling procedures). For this study we have chosen h=20, i.e. monthly returns. Monthly returns are in general approximately independent, while still providing a decent amount of data points (see autocorrelation analysis in chapter 4.2). The appropriateness this choice is confirmed by the results in Kaufmann’s study (Kaufmann, 2004).

In chapter 5 we investigated the performance of a number of empirical methods to scale VaR on simulated data. This chapter regards real fixed income log-returns from the data described in chapter 3 and 4. We have chosen to use the best performing empirical method (Bootstrap) and the square-root-of-time rule from chapter 5 as well as random walks with normal and student-t innovations and an AR-process to scale the 20-day VaR to 60 days (3 months) and 250 days (1 year). Using time series modeling in scaling VaR is a two step method, where one first estimate the parameters of the process in question for h-day returns, for example mean and standard deviation in the case of a random walk with normal innovations, and then simulate (i.e. monte-carlo simulation) a large number of K-day returns. One then obtains the scaled K-day VaR by taking the appropriate quantile of these returns.

In the case of the random walk method using normal innovations the analytical n-day VaR is as previously stated equal to the square-root-of-time rule. We also have an analytical solution to the n-day VaR of an AR(1)-process with normal innovations:

Lemma:

For an autoregressive model of order 1, AR(1), with normal innovations,

$$X_t = \varphi X_{t-1} + \epsilon_t \,, \quad \epsilon_t = \sim i.i.d.\, N(0, \sigma^2)$$

Both 1-day and n-day log-returns are normally distributed:

$$X_t \sim N\left(0, \frac{\sigma^2}{1-\varphi^2}\right) \quad and \quad \sum_{t=1}^{n} X_t \sim N\left(0, \frac{\sigma^2}{1-\varphi^2}\left(n - 2\varphi\frac{1-\varphi^n}{1-\varphi^2}\right)\right)$$

Hence, utilizing that $VaR_\alpha(X) = \sigma x_\alpha$ *, where* $X \sim N(0, \sigma^2)$ *and* x_α *is the* α*-quantile of the standard normal distribution, we get*

$$VaR^{(n)} = \sqrt{\frac{1+\varphi}{1-\varphi}(n - 2\varphi\frac{1-\varphi^n}{1-\varphi^2})}\, VaR^{(1)}$$

Using GARCH and AR-GARCH-processes to scale VaR were not applicable due to stationarity issues in certain periods of the data sets.

We proceed with a back-testing method to check the precision of the different methods. These results are then compared to those in chapter 5, given the distribution analysis of the index data in chapter 4.

6.1 Back-testing procedure

The difficulty with performing back-testing of quarterly and yearly VaR is the lack of sufficient return data for these periods in the indexes used. We therefore employed a method where we use rolling 1-month returns (i.e. the 1-month period at hand is moved 1 day forward in the next iteration) to estimate 1-month VaR and scale this to 3-month and 1-year VaR, and then we compared this VaR to the actual 3-month and 1-year (rolling) return. The number of exceedances are then counted and divided by the total number of comparisons to get the approximate probability of (in the case of quarterly VaR) the 3-month (future) return exceeding the 3-month VaR. In this case with 99% VaR a good result of this number would be one close to 0.01, i.e. satisfactory performance of the scaling method would imply that the number of exceedances would be close to 1% of the total number of comparisons.

The procedure in detail for 3-month VaR (analogous to 1-year VaR):

1. Daily data from day 1 to day i is used to estimate 99% 1-month VaR (i=length of data/2)
2. The VaR figure is scaled to 3-month (60 days) VaR with the respective methods
3. The 3-month VaR is compared to the actual 3-month return from i to i+60
4. i=i+1

This procedure was performed for all indices described in chapter 3, results given below.

6.2 Results OMRX-Bond Index

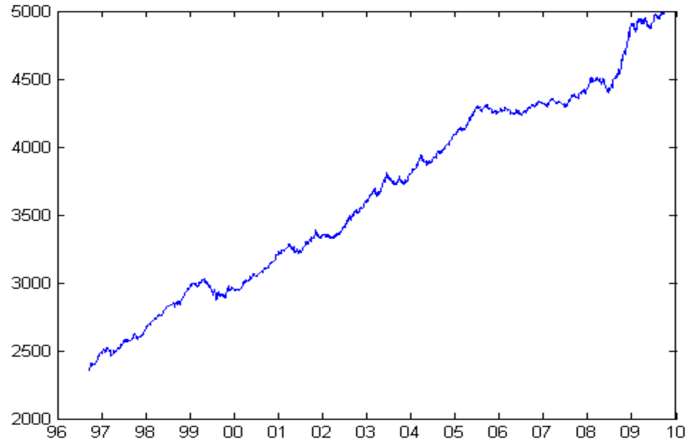


Figure 6.2.1: Price development OMRX-Bond Index

Yearly VaR

OMRX-Bond Index yearly VaR	
Scaling method	Exceedance ratio
Bootstrap	5.0%
Square-root-of-t	1.1%
Random Walk t innov.	5.6%
Random Walk norm innov.	6.3%
AR(1)-process	6.1%

Table 6.2.1: 99% 1-year VaR Exceedance ratios for OMRX-Bond Index

Given a good estimation procedure of 1-year VaR, the exceedance ratio, i.e. the percentage of yearly (rolling) returns exceeding the (rolling) 99% 1-year VaR, should be around 1%. In this case the square-root-of-time rule is clearly the best performing method with 1.1% exceedance ratio (see table 6.2.1 above). The other methods give a ratio of about 5-6%, hence underestimating the 1-year VaR of OMRX-Bond Index.

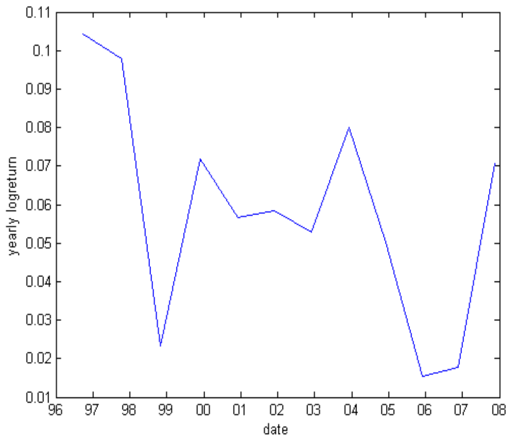


Figure 6.2.2: Yearly log-returns OMRX-Bond Index

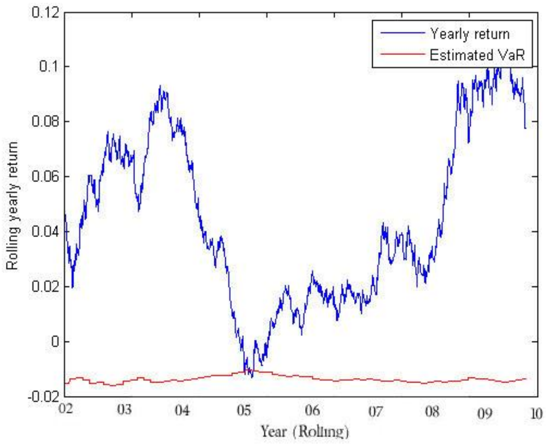


Figure 6.2.3: Actual rolling yearly return vs. corresponding 99% 1-year VaR estimated with square-root-of-time

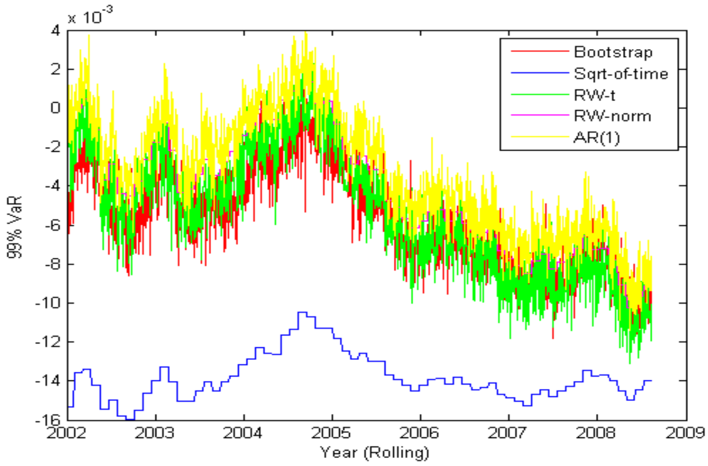


Figure 6.2.4: 99% 1-year VaR for OMRX-Bond Index with the different scaling procedures

Looking at the yearly log-returns in figure 6.2.2 we see that the return has varied between 10% and 1.5%, with large annual variations. The rolling (1-day steps) yearly returns in figure 6.2.3 are as explained in chapter 6.1 obtained from the second half of the total data, corresponding to years 2002 and onwards in figure 6.2.2. We see that the yearly VaR, in figure 6.2.3, estimated with the square-root-of-time rule, remains fairly constant although new data (more rolling years) are added to the estimation procedure. Note that we in these graphs and graphs following below plot the negative VaR, i.e. 'losses'. The exceedances, which are were the blue line (rolling yearly return) is below the red line (1-year rolling VaR estimate), occur around the period 2006-2007. The underestimation of the 1-year VaR by all the methods is probably due to the fact that the lowest returns occur at the end of the data set, thus these returns are not included in the estimation of VaR corresponding to the period around 2006-2007. Looking at figure 6.2.4 wee see that all methods behave similarly, square-root-of-time being slightly more conservative and having less volatile outcomes than the other methods, confirming the results of chapter 5.

Quarterly VaR

OMRX-Bond Index 3m VaR	
Scaling method	Exceedance ratio
Bootstrap	0.5%
Square-root-of-t	0.2%
Random Walk t innov.	0.4%
Random Walk norm innov.	0.6%
AR(1)-process	0.6%

Table 6.2.2: 99% 3-month VaR Exceedance ratios for OMRX-Bond Index

When backtesting the quarterly VaR for OMRX-Bond Index random walk with normal innovations and the AR(1)-process scaling procedures came closest to a 1% exceedance ratio, both with 0.6%. All methods have exceedance ratios below 1%, indicating that they tend to overestimate the 3-month VaR in this case.

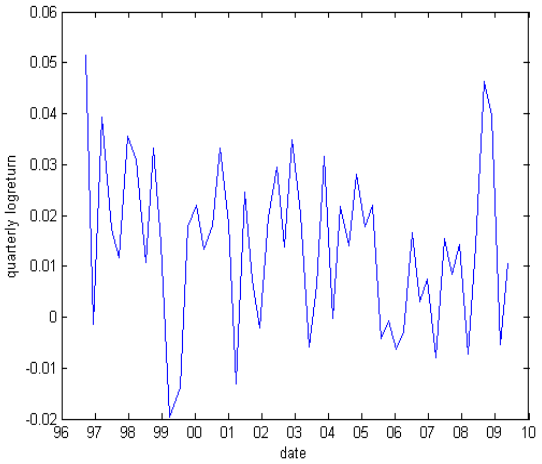


Figure 6.2.5: Quarterly log-returns OMRX-Bond Index

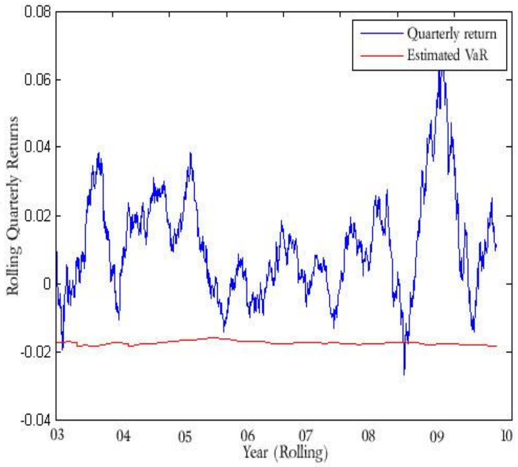


Figure 6.2.6: Actual rolling quarterly return vs. corresponding 99% 3-month VaR estimated with random walk (normal innovations)

Looking at the quarterly log-returns for OMRX-Bond Index in figure 6.2.5, we see them varying between 5% down to -2% in 1999. Comparing these to figure 6.2.6 (rolling quarterly return and corresponding estimated VaR from 2003 and onwards), we see that the exceedances, i.e. where the rolling quarterly log-returns is below the 3-month 99% corresponding VaR, occur when the return is around -2%. The slight overestimation of the VaR by all the methods seen in table 6.2.2 is probably due to the large negative returns in the beginning of the data set around 1999.

6.3 Results JPMorgan Em. Mark. Bond Index

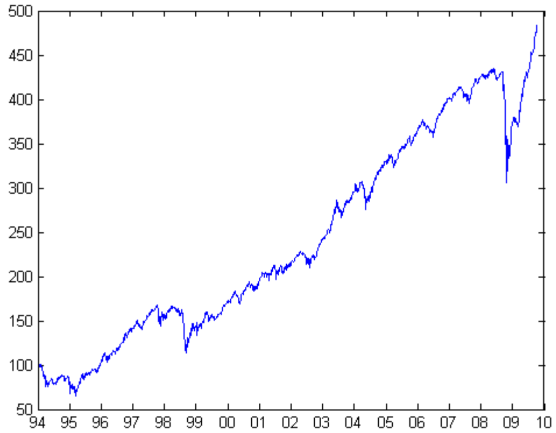


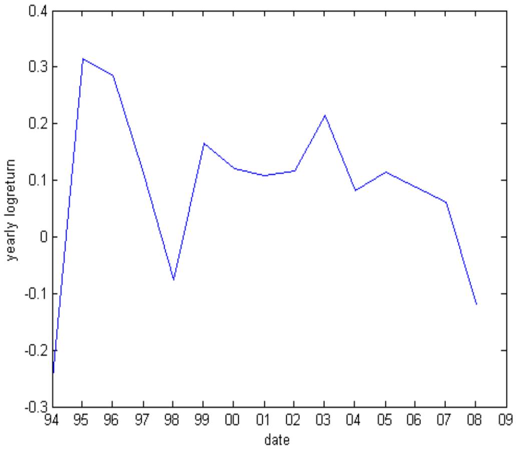
Figure 6.3.1: Price development JPMorgan Em. Mark. Bond Index

Yearly VaR

JPMorgan Em. Mark. Bond Index yearly VaR	
Scaling method	Exceedance ratio
Bootstrap	0.3%
Square-root-of-t	0.0%
Random Walk t innov.	0.8%
Random Walk norm innov.	0.8%
AR(1)-process	0.8%

Table 6.3.1: 99% 1-year VaR Exceedance ratios for JPMorgan Em. Mark. Bond Index

The random walk-procedures as well as the AR(1)-procedure performs equally well in the case of 99% 1-year VaR for JPMorgan Em. Mark. Bond Index as seen in table 6.3.1 above, with exceedance ratios of 0.8%, i.e. a slight overestimation of the VaR. Bootstrap and square-root-of-time overestimates the VaR.



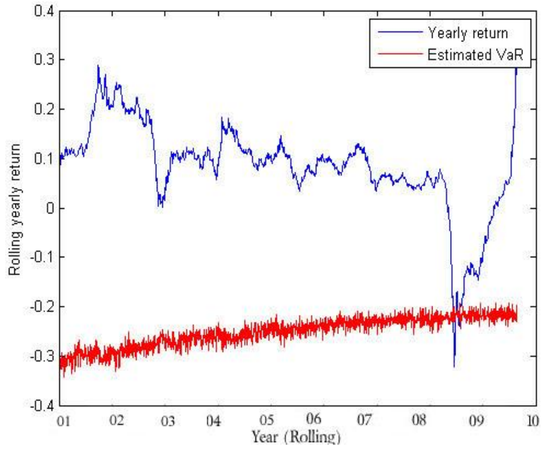


Figure 6.3.2: Yearly log-returns JPMorgan Em. Mark. Bond Index

Figure 6.3.3: Actual rolling yearly return vs. corresponding 99% 1-year VaR estimated with random walk (t(3)-innovations)

Looking at the yearly log-returns in figure 6.3.2, we see a large negative return in 1994 of about -20%, with another large dip in 2008 of about -10%. The effect of this on the estimated yearly VaR is clear when looking at figure 6.3.3 (returns and VaR from 2001 and onwards); the initial dip in the data set around 1994 gives a low VaR of around -30% which then steadily increases as additional rolling returns are on the positive side. The post-Lehmann dip in 2008, with some rolling yearly returns below -30%, is where the exceedances occur. The overestimation of VaR by all methods is most likely due to the large negative returns in the very beginning of the data set.

Quarterly VaR

JPMorgan Em. Mark. Bond Index 3m VaR	
Scaling method	Exceedance ratio
Bootstrap	1.3%
Square-root-of-t	1.0%
Random Walk t innov.	2.4%
Random Walk norm innov.	2.4%
AR(1)-process	2.4%

Table 6.3.2: 99% 3-month VaR Exceedance ratios for JPMorgan Em. Mark. Bond Index

As opposed to the yearly VaR, Bootstrap and square-root-of-time performs well in scaling quarterly VaR for JPMorgan Em. Mark. Bond Index. Square-root-of-time performs excellent with 1.0% exceedance ratio, while the other methods underestimate the 3-month VaR.

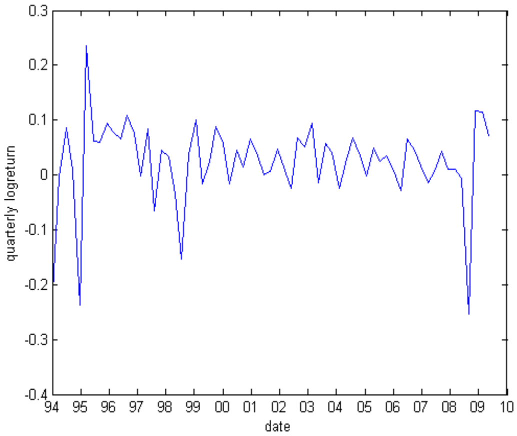


Figure 6.3.4: Quarterly log-returns JPMorgan Em. Mark. Bond Index

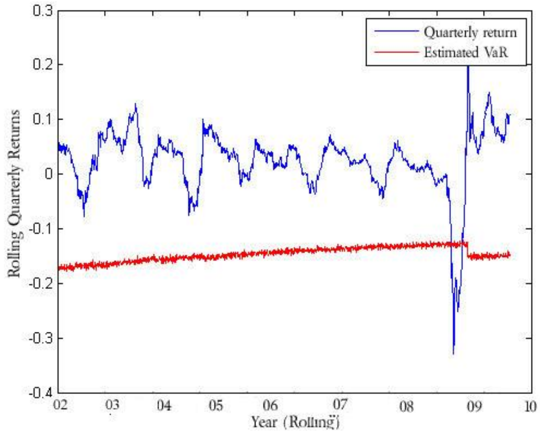


Figure 6.3.5: Actual rolling quarterly return vs. corresponding 99% 3-month VaR estimated with an AR(1)-process (normal innovations)

Given figure 6.3.4 above, we see large negative quarterly returns of around -20% in the beginning of the data set around 1994/1995. Following this is a period of relatively low volatility, the majority of returns being on the positive side, followed by a large dip in 2008. This is reflected in the estimation of quarterly VaR in figure 6.3.5, beginning around 2002. The VaR is initially low around -18%, steadily increasing due to following years having positive returns. The exceedances occur in the large dip around 2008, due to the fact that this dip is larger than the dips used to estimate VaR in the beginning of the data. We also see a small downward correction in the VaR after this period of low rolling returns.

6.4 Results FTSE Euro Corp. Bond Index

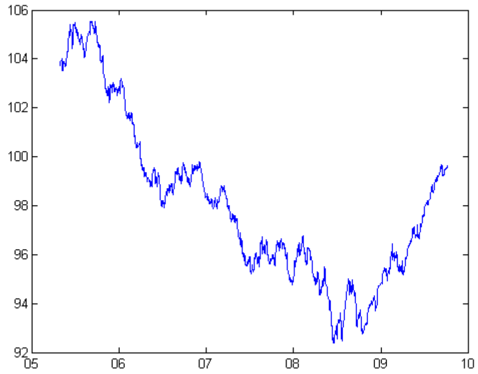


Figure 6.4.1: Price development FTSE Euro Corp. Bond Index

Yearly VaR

FTSE Euro Corp. Bond Index yearly VaR	
Scaling method	Exceedance ratio
Bootstrap	0.0%
Square-root-of-t	0.0%
Random Walk t innov.	0.0%
Random Walk norm innov.	0.0%
AR(1)-process	0.0%

Table 6.4.1: 99% 1-year VaR Exceedance ratios for FTSE Euro Corp. Bond Index

There are no exceedances in the backtesting procedure for FTSE Euro Corp. Bond Index. This is an implication of the data set being rather small, only 4 years of returns, as well as the intial first half of the data, which is the one used for estimating VaR, having a downward slope (see figure 6.4.1 above) followed by an upward slope. The rolling returns of the first half are hence to a majority negative, which induces a very negative estimation of the VaR. This low VaR is thus never exceeded in the second half of the data where returns are mostly on the positive side.

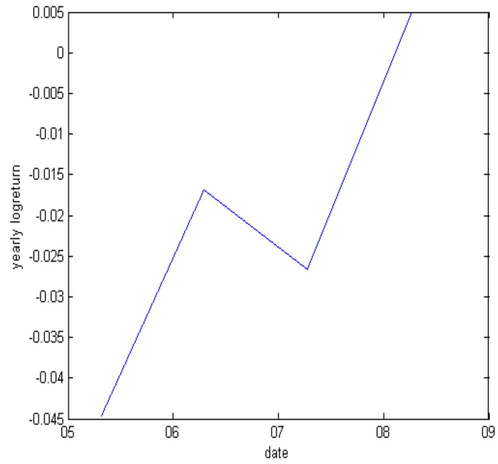


Figure 6.4.2: Yearly log-returns FTSE Euro Corp. Bond Index

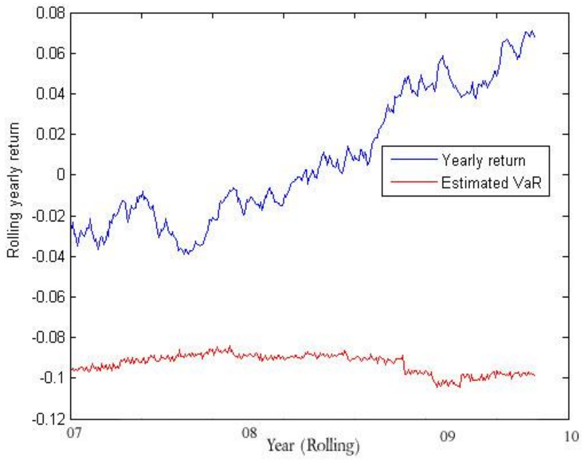


Figure 6.4.3: Actual rolling yearly return vs. corresponding 99% 1-year VaR estimated with bootstrap

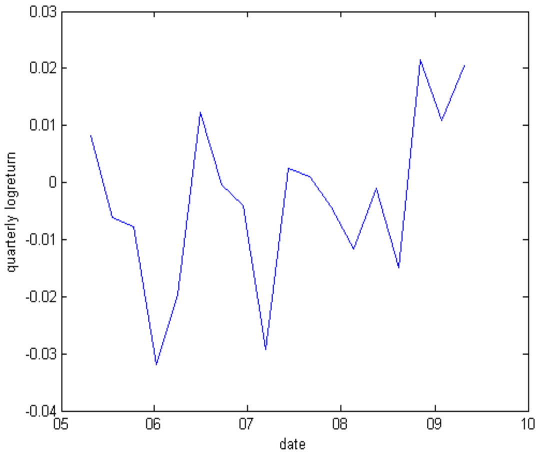
The low yearly returns in the first half of the data induce a VaR of around -10%. The following years of mostly positive returns thus never exceed the yearly VaR, which we see in figure 6.4.3.

Quarterly VaR

FTSE Euro Corp. Bond Index 3m VaR	
Scaling method	Exceedance ratio
Bootstrap	0.1%
Square-root-of-t	1.0%
Random Walk t innov.	0.0%
Random Walk norm innov.	0.0%
AR(1)-process	0.0%

Table 6.4.2: 99% 3-month VaR Exceedance ratios for FTSE Euro Corp. Bond Index

As we see in table 6.4.2, square-root-of-time performs well for quarterly VaR in the case of FTSE Euro Corp. Bond Index, with an exceedance ratio of 1.0%. This is noticeable, given that the square-root-of-time often is seen as a conservative measure supposed to overestimate the VaR. Here we have the opposite outcome, where the other methods overestimate the quarterly VaR.



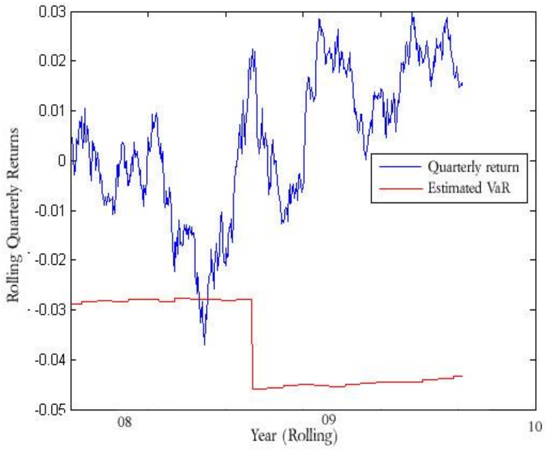


Figure 6.4.4: Quarterly log-returns FTSE Euro Corp. Bond Index

Figure 6.4.5: Actual rolling quarterly return vs. corresponding 99% 3-month VaR estimated with square-root-of-time

Looking at figures 6.4.4 and 6.4.5, we see that the first half of the data, with two quarterly returns around -3%, implies a VaR of around -2.8%. The exceedances occur in the end of 2008, where rolling returns are down to around -3.5%. The 3-month VaR, incorporating these exceedances, is then corrected sharply down to -4.7%.

6.5 Results Dow Jones Corp. Bond Index

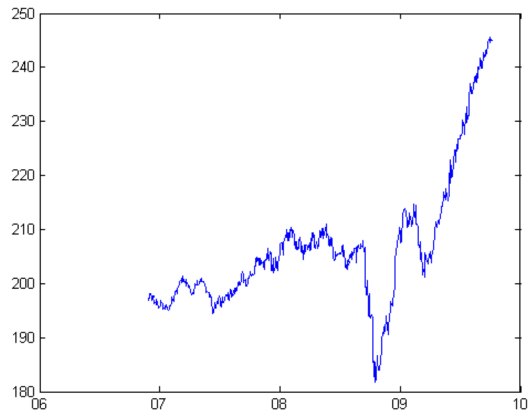


Figure 6.5.1: Price development Dow Jones Corp. Bond Index

Yearly VaR

Dow Jones Corp. Bond Index yearly VaR	
Scaling method	Exceedance ratio
Bootstrap	0.0%
Square-root-of-t	0.0%
Random Walk t innov.	0.0%
Random Walk norm innov.	0.0%
AR(1)-process	0.0%

Table 6.5.1: 99% 1-year VaR Exceedance ratios for Dow Jones Corp. Bond Index

As in the case of yearly returns for FTSE Euro Corp. Bond Index there are no exceedances when backtesting yearly VaR for Dow Jones Corp. Bond Index, also here due to lack of return data (only three years) as well as initial rolling returns being low relative to later rolling returns.

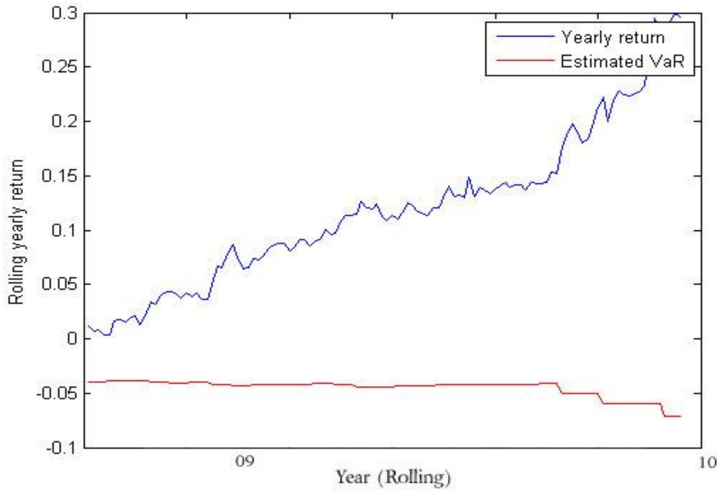


Figure 6.5.2: Actual rolling yearly return vs. corresponding 99% 1-year VaR estimated with random walk (normal innovations)

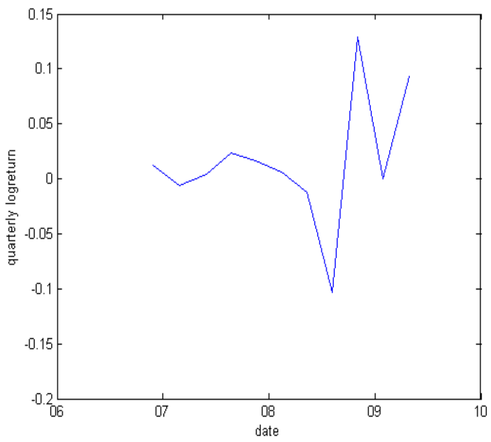
Yearly returns where +4% in 2007 and -5% in 2008. We end up with a period of about one year to utilize for checking whether our rolling estimates of the yearly VaR is exceeded by the following actual yearly (rolling) returns. In this case, looking at the price development in figure 6.5.1, we have an upward slope in the latter part of the data set, implying, as we see in figure 6.5.2, that there are no exceedances.

Quarterly VaR

Dow Jones Corp. Bond Index 3m VaR	
Scaling method	Exceedance ratio
Bootstrap	18.2%
Square-root-of-t	22.0%
Random Walk t innov.	17.2%
Random Walk norm innov.	17.6%
AR(1)-process	17.6%

Table 6.5.2: 99% 3-month VaR Exceedance ratios for Dow Jones Corp. Bond Index

Table 6.5.2 paints a picture completely different from the 1-year VaR case. Instead of no exceedances we have very high exceedance ratios for the quarterly VaR of around 17%, i.e. a severe underestimation of the VaR. As in the case of the yearly VaR for Dow Jones Corp. Bond Index we have a somewhat limited data set, which makes our predictions unstable.



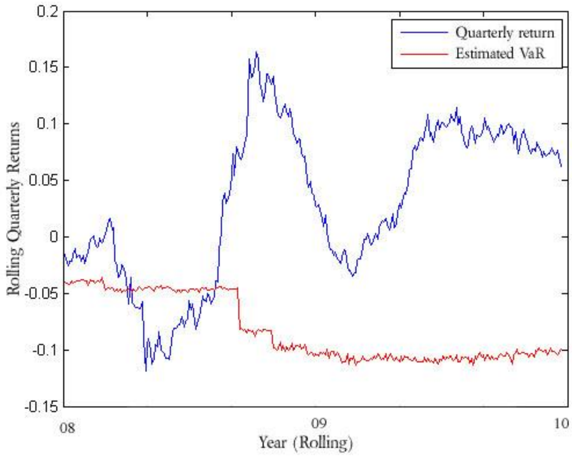


Figure 6.5.4: Quarterly log-returns Dow Jones Corp. Bond Index

Figure 6.5.5: Actual rolling quarterly return vs. corresponding 99% 3-month VaR . estimated with random walk (t(6))

Looking at figure 6.5.4, we see that yearly returns of the first half of the data are close to zero. We obtain a VaR estimate of around -4%, which is then exceeded in the post-Lehmann period in the end of 2008, where quarterly returns drop down to -10%.

6.6 Results Credit Suisse High Yield Index

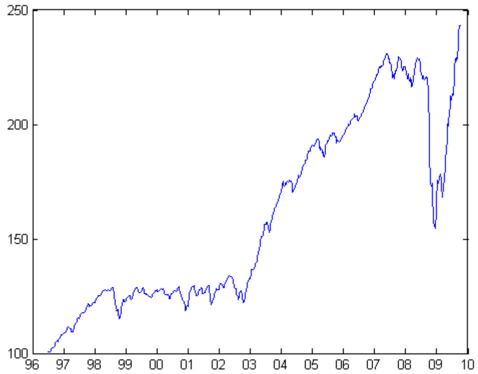


Figure 6.6.1: Price development Credit Suisse High Yield Index

Yearly VaR

Credit Suisse High Yield Index yearly VaR	
Scaling method	Exceedance ratio
Bootstrap	12.6%
Square-root-of-t	11.9%
Random Walk t innov.	12.6%
Random Walk norm innov.	13.0%
AR(1)-process	13.0%

Table 6.6.1: 99% 1-year VaR Exceedance ratios for Credit Suisse High Yield Index

Exceedance ratios for the yearly VaR of Credit Suisse High Yield Index are significantly higher than 1%, around 12% for all scaling methods, i.e. all methods underestimate the yearly VaR. Square-root-of-time performs least worse with a ratio of 11.9%.

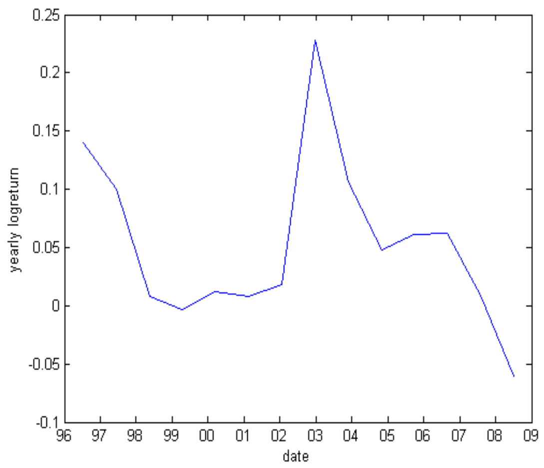


Figure 6.6.2: Yearly log-returns Credit Suisse High Yield Index

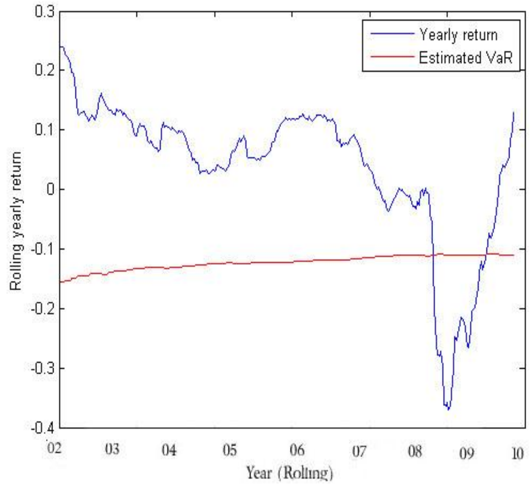


Figure 6.6.3: Actual rolling yearly return vs. corresponding 99% 1-year VaR estimated with square-root-of-time

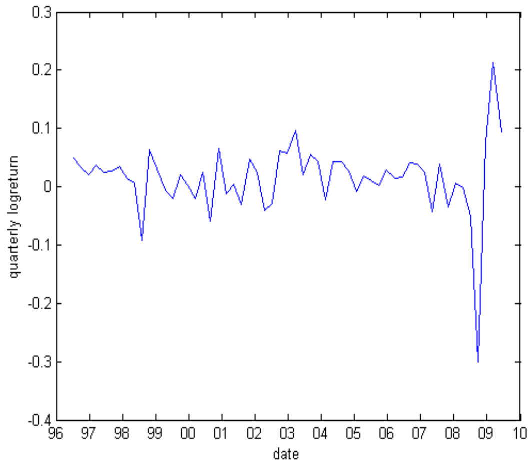
Given figure 6.6.2 we see that yearly returns are highly volatile, ranging from +25% in 2002 to -5% in 2008. The large post-Lehmann drop in the end of 2008 (seen in the price development in figure 6.6.1) is where the exceedances occur. The rolling yearly returns of the first half of the data gives us an estimate of the yearly VaR of around -15% (see figure 6.6.3), which is then severely exceeded in the end of 2008 with rolling yearly returns being as low as -35%.

Quarterly VaR

Credit Suisse High Yield Index 3m VaR	
Scaling method	Exceedance ratio
Bootstrap	5.1%
Square-root-of-t	4.8%
Random Walk t innov.	5.1%
Random Walk norm innov.	5.1%
AR(1)-process	5.1%

Table 6.6.2: 99% 3-month VaR Exceedance ratios for Credit Suisse High Yield Index

As in the case for Credit Suisse High Yield Index with yearly VaR, we get an underestimation of VaR also in the quarterly estimations with exceedance ratios being close to 5% for all scaling methods. As with yearly VaR, square-root-of-time performs best with an exceedance ratio of 4.8%, solidifying its reputation as the more conservative measure (although in the case with Credit Suisse High Yield Index not conservative enough).



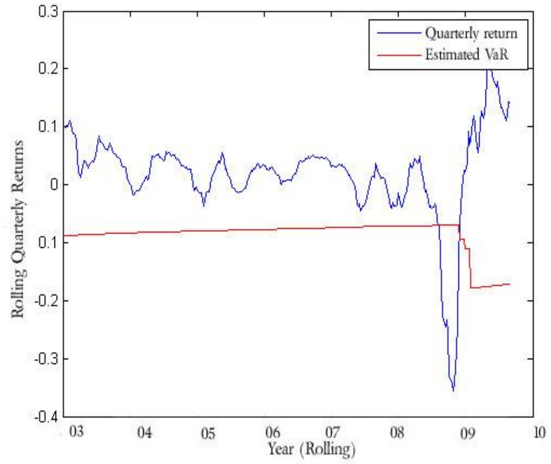


Figure 6.6.4: Quarterly log-returns Credit Suisse High Yield Index

Figure 6.6.5: Actual rolling quarterly return vs. corresponding 99% 3-month VaR . . estimated with square-root-of-time

Quarterly returns remain relatively stable, varying between -10% and +10% up to the end of 2008. In figure 6.6.5 this is reflected in the estimation of the quarterly VaR, remaining relatively constant during periods corresponding to years 2003 to 2007 with a sharp correction after the dip in late 2008. As in the case of yearly VaR, quarterly VaR is underestimated since the returns before 2008 are does not contain any period of the same negative magnitude.

6.8 Scaling of 1-day to 250-day VaR without Intermediary Step

Simply scaling 1-day VaR to 60-day or 250-day VaR is not a viable alternative since it produces too uncertain results. This is partly due to a significant dependence between daily returns (see chapter 4.2) in the case of random-walk methods. Also, other characteristics (e.g. leptokurtosis) of 1-day return distributions might not be the same as in say 250-day return distributions. 1-day and 250-day returns are simply not comparable; their parameters depend on completely different factors. For example, daily returns are highly dependent of the behavior of investors, and hence to a large part affected by the psychology of the individual. Yearly returns on the other hand should be more correlated to macroeconomic factors. We illustrate this by examining the effects of scaling 1-day VaR directly to 250-day VaR for OMRX-Bond Index below.

Yearly VaR with 1-day to 250-day scaling

OMRX-Bond Index yearly VaR (scaled from 1-day returns)	
Scaling method	Exceedance ratio
Bootstrap	9.4%
Square-root-of-t	2.5%
Random Walk t innov.	9.4%
Random Walk norm innov.	9.4%
AR(1)-process	9.5%

Table 6.8.1: 99% 1-year VaR Exceedance ratios for OMRX-Bond Index, using 1-day to 250-day scaling

Clearly, scaling 1-day VaR directly to 250-day VaR, skipping the intermediate step of using estimated 1-month VaR, produces more unreliable results. Square-root-of-time performs less worse than the other methods with an exceedance ratio of 2.5%, however significantly worse than with the intermediate step in table 6.2.2 (exceedance ratio 1.1%). This is as discussed earlier most likely a function of the daily and yearly return distributions being too different, i.e. they do not share basic distribution characteristics, thereby making scaling of the quantiles difficult.

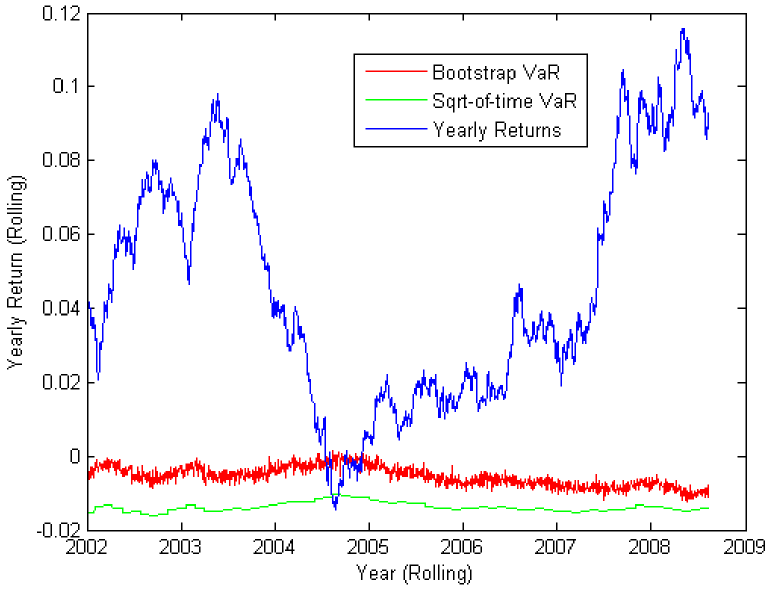


Figure 6.8.1: Actual rolling yearly return vs. corresponding 99% 1-year VaR estimated with square-root-of-time and bootstrap

In figure 6.8.1 one we see why square-root-of-time outperforms bootstrap so significantly, when comparing exceedances in table 6.8.1 above. Since square-root-of-time is slightly more conservative than bootstrap and the other methods, as we showed in chapter 5 on simulated data as well, we get several times more exceedances. This is of course dependent of the data set in question and not a statistical truth regarding the relative performance of square-root-of-time and the other methods. The conclusion that can be drawn here is that square-root-of-time is a conservative measure relative to the other methods.

6.3 Summary of Results and Discussion

Results per scaling method

Bootstrap:
For the more stationary series OMRX-Bond Index, JPMorgan Em. Mark. Bond Index and FTSE Euro Corp. Bond Index the bootstrap scaling method has more exceedances than the square-root-of-time rule but less than the remaining methods, i.e. it is less conservative than square-root-of-time and slightly more than the time series methods, for these indicies. Inconclusive results for Dow Jones Corp. Bond Index and Credit Suisse High Yield Index. Performance is not significantly different from the random walk and AR methods.

Square-root-of-time:
Excellent performance for yearly VaR in the case of OMRX-Bond Index and quarterly VaR for JPMorgan Em. Mark. Bond Index and FTSE Euro Corp. Bond Index with exceedances very close to the desired 1%. On the conservative side relative to the other methods in the majority of cases with the exception of quarterly VaR for FTSE Euro Corp. Bond Index and Dow Jones Corp. Bond Index, where it actually is the least conservative method. This unexpected effect could well be an implication of the lack of data in these indices.

Random Walk student-t innovations:
Best relative performance for quarterly VaR in Dow Jones Corp. Bond Index, although still severely underestimating the VaR like the other methods. Performance is otherwise on the less conservative side like the other time series-model methods.

Random Walk normal innovations:
Good performance for quarterly VaR in OMRX-Bond Index with a slightly conservative exceedance ratio of 0.6%. Performance is otherwise on the less conservative side like the other time series-model methods.

AR(1) :
Like random walk with normal innovations 0.6% exceedance ration for quarterly VaR in OMRX-Bond Index, otherwise on the less conservative side like the other time series-model methods.

When observing the results of the different scaling procedures, the square-root-of-time rule performs extraordinary well in the case of yearly VaR in OMRX-Bond Index, quarterly VaR for JPMorgan Em. Mark. Bond Index and quarterly VaR for FTSE Euro Corp. Bond Index. The other methods are roughly equal. All methods have a tendency to overestimate VaR with an exceedance ratio below 1% for the more stationary OMRX-Bond Index, JPMorgan Em. Mark. Bond Index and FTSE Euro Corp. Bond Index, while severely underestimating it for Dow Jones Corp. Bond Index and Credit Suisse High Yield Index. Comparing these results to those in chapter 5 is difficult, but one clear tendency which we saw in chapter 5 is that of good performance relative to the other methods in the case of the square-root-of-time rule with simulated data. This is also confirmed in this chapter for real fixed income data. As we saw in the distribution analysis of chapter 4, monthly returns for OMRX-Bond Index and FTSE Euro Corp. Bond Index are closer to the normal distribution, without the pronounced fat-tails of the other indices. Considering this and the high degree of independence between monthly returns seen in the autocorrelation analysis of chapter 4.2, and the fact that the square-root-of-time rule is the analytical solution to scaling quantiles of a random walk with normal innovations, the satisfactory results of square-root-of-time are not surprising. The relatively good results for the more fat-tailed indices JPMorgan Em. Mark. Bond Index is probably due to the conservative nature of the square-root-of-time rule.

Given our backtesting analysis, scaling 99% VaR works well for the less volatile indices OMRX-Bond Index and FTSE Euro Corp. Bond Index as well as well as the fat-tailed JPMorgan Em. Mark. Bond Index. Exceedances are reasonably close to 1% both in the case of yearly and quarterly VaR. For Dow Jones Corp. Bond Index we have the contradictory results of zero exceedances in the case of yearly VaR, i.e. overestimation of the VaR, and severe underestimation of the VaR in the quarterly dito with exceedances around 20%. For Credit Suisse High Yield Index we have significant underestimation of the VaR for both yearly and quarterly estimations. Looking at the price development of the data, one can see that OMRX-Bond Index, FTSE Euro Corp. Bond Index and JPMorgan Em. Mark. Bond Index have a relatively smooth development without the sudden large drops in the other indices, implying a more stationary process. This is likely why scaling of the quantiles works better for these data sets. The limited number of return data for FTSE Euro Corp. Bond Index and Dow Jones Corp. Bond Index also gives somewhat unreliable results, possibly explaining the deviances.

The worst possible scenario has not happened yet. This simple fact is evident when backtesting the VaR of these fixed income indices. What we see in the majority of cases is that the 99% VaR estimated from previous data is exceeded during the period after the investment bank Lehmann Brothers went bankrupt in October 2008. For the more credit worthy assets OMRX-Bond Index and FTSE Euro Corp. Bond Index the sharp decline in asset price is not as pronounced as for the more high-yielding indices; the low yearly and quarterly rolling returns exceeding the 99% scaled VaR of these periods roughly equal a one in a hundred event given the length of our data sets, enabling us to get a good estimate of the VaR. However, the post-Lehmann crisis affected more risky fixed income assets such as Dow Jones Corp. Bond Index and Credit Suisse High Yield Index very negatively, giving us yearly and quarterly returns far below what was seen in the earlier parts of the data. Because of this the VaR for the post-Lehmann period is underestimated. One can conclude that when estimating longer period VaR for risky assets with significant credit risk such as high-yielding bonds, one should be well on the conservative side. The square-root-of-time rule is conservative in most cases as we have shown, yet not enough and not always. Using the scaling procedure presented in this chapter, with carefully chosen scaling method and respective parameters depending on asset class, is however a good alternative for fixed income assets with limited credit risk such as government bonds and European investment grade bonds.

7. Measuring Risk in a Non-Stationary Portfolio

In this chapter we investigate the implications of a non-fixed portfolio allocation to risk management. The results from chapters 5 and 6 are based on the presumption that the portfolio allocation remains constant over the investment horizon. Though a reasonable assumption in the case of scaling 1-day VaR to 10-day VaR, this is clearly not the case when looking at time horizons of one month and more in actively managed portfolios. The portfolio managers will during the period of risk measuring react to market events, thus changing the portfolio allocation and hence the distribution of returns. Incorporating this behavior into a quantitative model for obtaining risk measures such as VaR is of course very difficult, considering that the portfolio managers investment decisions are subject to the psychology of the individual. However, when portfolio allocation decisions (i.e. the weights of the different asset classes) are directed completely by rules set up in the investment policy this becomes possible. One example of such a strategy is CPPI (Constant Proportion Portfolio Insurance), a way of guaranteeing a certain percentage of the total portfolio value by changing the weights between risky and risk-free assets in response to market movements.

CPPI is widely used in portfolio hedging, its main benefits versus e.g. option hedging being its low cost and ease of implementation, the drawback being an event risk as in the case of sudden large price movements in the market.

We will use a model for determining 3-month VaR of this portfolio which uses the bootstrap scaling method described in chapter 5 to scale 1-month VaR to 3-month VaR. Scenarios for the returns of the different assets are derived from appropriate risk-factors such as various indices. This model provided us with a practical way to implement and test the effect of CPPI on the VaR of the portfolio, allowing us to compare the risk before and after using CPPI.

7.1 Constant Proportion Portfolio Insurance (CPPI)

All of the scaling methods in the previous chapters assume that the portfolio allocations remain constant over the relevant time horizon. This is seldom a realistic assumption, for instance when looking at a time horizon of three months. When relaxing the constraint of a fixed portfolio allocation one need to have a set of predetermined allocation rules or management actions, i.e. reactions to market performance. In the light of new regulations in the insurance business where companies' internal models might be allowed to incorporate management actions (CEIOPS' Advice for level 2 Implementing Measures on Solvency II: Technical Provisions – Assumptions about Future Management Actions, 2009) we implement such a structure in the model; CPPI.

The CPPI was first introduced for fixed income instruments by Perold (1986) and for equities by Black and Jones (1987). It is a quite simple strategy allowing its user to dynamically allocate assets over time. Two parameters are exogenous to CPPI, a floor and a multiple. The floor is the minimum amount that you do not want the portfolio performing below, e.g. 90 per cent of the portfolios start value. The multiple is chosen as the inverse of the assumed gap risk where the gap risk is the assumed maximum drop one might experience in the risky asset in your portfolio during an intra-period. Handelsbankens Asset Management division has for instance modeled the intraday gap risk to be ca -12% for a risky asset such an ETF with an equity index as the underlying. In this case we would therefore get a multiple of ca 8 ($\sim 1/0.12$). In the literature one often uses slightly smaller multiples, such as 4 or 5 (Black & Perold p. 409 1992, Ameur & Prigent p. 3 2006).

Once a floor and multiple are chosen the allocations can be determined by first calculating the so called cushion as the difference of the current portfolio value (risky and riskless assets) and the floor. The amount allocated to the risky asset is then simply the cushion multiplied by the multiple and the amount allocated to the riskless asset is what is left of the portfolio. After one period a new cushion is calculated. The new cushion will be larger if the risky asset has had a positive return and it will be smaller if it has performed a negative return and since the multiple (in our case) does not change over time, the allocation to the risky asset will increase or decrease accordingly. As long as the intra-period performance is not worse than the assumed gap risk, the portfolio will never fall below its floor value. Of course, several possible extensions to CPPI comes to mind, for instance that the floor grows with the rate of the risk free interest rate (this is actually the original structure), where the floor may be adjusted to lock-in good performance, where the multiple varies over time (Ameur & Prigent, 2006), where the exposure to risky asset is not allowed to exceed a certain proportion of the portfolio or where you are not allowed to short the riskless asset.

The CPPI is well suited to be implemented as a management action strategy in an internal risk model since all allocations are known given a market performance and since the implementation is not computationally heavy.

7.2 Implementation of CPPI

We have chosen eleven portfolios that will have a CPPI structure implemented for a portion of their equity holdings. The CPPI model will have a fixed floor (i.e. not growing with the riskless interest rate) that is 90 per cent of the NAV at a certain time, but the floor may be changed discretionary, for instance if you want to lock in a good performance. Models, which will not be dealt with here, have given a gap risk of 12 per cent per day and thus a multiplier of ca 8. All portfolios have a maximum proportion that equities may have of the entire portfolio, ranging from 3 per cent to 23 per cent.

The original (stationary) model is based on historical data with a period of one month. The allocations have been mapped to 33 risk factors for which we have ca 300 months of historical data. Bootstrap is then used and for the stationary model we get a three month return by aggregating three bootstrapped months, thus the independence structure is preserved within a month. The most desirable would be to change this to a period of one day since this is the rebalancing period available. However such a drastic change to the existing model lies beyond the scope of this paper and hence we will keep the one month period. This means that the portfolios will be re-allocated two times, i.e. between the first and the second month and between the second month and the third month to give a corresponding risk measure.

The change from one day periods to one month periods means that the gap risk assumption of 12 per cent is no longer valid. We will use a set of gap risks giving multiplies ranging from 3 to 8. Please also note that only a proportion of the portfolios are allocated to the CPPI structure, hence not all risky assets are protected by the CPPI structure (nor are all equity holding). This means that non-favorable moves in interest rates that dominate the portfolios might also dominate the CPPI structure.

7.3 Results and discussion

Here we present the results from an implementation of the CPPI in the existing stationary model. First we present two tables where equities either increase by five per cent each month or decreases by five per cent while all other assets have a zero return. These two tables are meant to show the reader the behavior and characteristics of CPPI in a more tidily manner than when including a varying performance of the other risky assets.

3-month returns (Stock return = -5 % per month, all else zero return)					
Portfolio	Ceiling (of entire sub-portfolio)	Proportion Allocated to CPPI	without CPPI	CPPI, multiplier=8	CPPI, multiplier=3
1	6%	4.03%	-1.1021%	-0.9382%	-0.8263%
2	6%	3.72%	-1.1606%	-0.9985%	-0.8883%
3	5%	3.24%	-0.9294%	-0.7976%	-0.7077%
4	3%	2.18%	-0.8248%	-0.7257%	-0.6579%
5	12%	8.82%	-2.5464%	-2.2167%	-1.9914%
6	12%	7.72%	-2.6166%	-2.2808%	-2.0508%
7	11%	8.05%	-2.3058%	-2.0187%	-1.8230%
8	8%	5.40%	-1.8272%	-1.5899%	-1.4276%
9	6%	4.13%	-1.2398%	-1.0759%	-0.9638%
10	12%	8.21%	-2.6856%	-2.3470%	-2.1148%
11	23%	16.91%	-3.9818%	-3.3538%	-2.9261%

Table 7.3.1: 3-month portfolio returns when equities value declines five per cent per month and all other assets have a zero return each month

3-month returns (Stock return = +5 % per month, all else zero return)					
Portfolio	Ceiling (of entire sub-portfolio)	Proportion Allocated to CPPI	without CPPI	CPPI, multiplier=8	CPPI, multiplier=3
1	6%	4.03%	1.2180%	1.3596%	1.0234%
2	6%	3.72%	1.2826%	1.4231%	1.0904%
3	5%	3.24%	1.0272%	1.1408%	0.8707%
4	3%	2.18%	0.9116%	0.9979%	0.7938%
5	12%	8.82%	2.8142%	3.1002%	2.4226%
6	12%	7.72%	2.8918%	3.1832%	2.4927%
7	11%	8.05%	2.5483%	2.7842%	2.2076%
8	8%	5.40%	2.0194%	2.2255%	1.7375%
9	6%	4.13%	1.3701%	1.5122%	1.1754%
10	12%	8.21%	2.9680%	3.2634%	2.5654%
11	23%	16.91%	4.4006%	4.9364%	3.6555%

Table 7.3.2: 3-month portfolio returns when equities value increases five per cent per month and all other assets have a zero return each month

3-month returns (bootstrap)								
Portfolio	Ceiling (of entire sub-portfolio)	Proportion Allocated to CPPI	without CPPI	CPPI multiplier=8	CPPI multiplier=3	without CPPI 99,5-quantile of returns	99,5-quantile of returns, CPPI multiplier=8	99,5-quantile of returns, CPPI multiplier=3
1	6%	4.03%	0.2750%	0.2786%	0.2371%	-2.6738%	-2.2596%	-2.1768%
2	6%	3.72%	0.2887%	0.2922%	0.2512%	-2.8021%	-2.3845%	-2.2964%
3	5%	3.24%	0.2615%	0.2643%	0.2311%	-2.2426%	-1.9367%	-1.8818%
4	3%	2.18%	0.2481%	0.2503%	0.2255%	-2.1073%	-1.9151%	-1.8646%
5	12%	8.82%	0.7337%	0.7390%	0.6462%	-7.1138%	-6.4199%	-6.3481%
6	12%	7.72%	0.7938%	0.7992%	0.7045%	-7.8080%	-7.4234%	-7.3567%
7	11%	8.05%	0.7120%	0.7158%	0.6359%	-6.6597%	-6.3041%	-6.2481%
8	8%	5.40%	0.4850%	0.4888%	0.4220%	-4.8247%	-3.8358%	-3.8089%
9	6%	4.13%	0.3089%	0.3125%	0.2709%	-2.9685%	-2.5415%	-2.4583%
10	12%	8.21%	0.7427%	0.7482%	0.6525%	-7.3175%	-6.4484%	-6.4247%
11	23%	16.91%	0.9957%	1.0066%	0.8352%	-10.2634%	-8.7598%	-8.2689%

Table 7.3.3: 3-month portfolio returns with bootstrapped returns for all assets. The three middle columns contain the mean returns and the three columns farthest to the right contain the 99.5-quantile of the returns.

standard deviation of 3-month returns (bootstrap)					
Portfolio	Ceiling (of entire sub-portfolio)	Proportion Allocated to CPPI	without CPPI	CPPI, multiplier=8	CPPI, multiplier=3
1	6%	4.03%	0.7035%	0.7345%	0.6052%
2	6%	3.72%	0.7405%	0.7709%	0.6426%
3	5%	3.24%	0.6406%	0.6640%	0.5650%
4	3%	2.18%	0.5954%	0.6119%	0.5404%
5	12%	8.82%	2.0424%	2.1094%	1.8231%
6	12%	7.72%	2.2227%	2.2873%	2.0125%
7	11%	8.05%	1.9212%	1.9747%	1.7343%
8	8%	5.40%	1.3435%	1.3965%	1.1670%
9	6%	4.13%	0.7927%	0.8234%	0.6932%
10	12%	8.21%	2.0965%	2.1678%	1.8624%
11	23%	16.91%	2.8432%	2.9734%	2.4341%

Table 7.3.4: The standard deviation of 3-month returns with bootstrapped scenarios

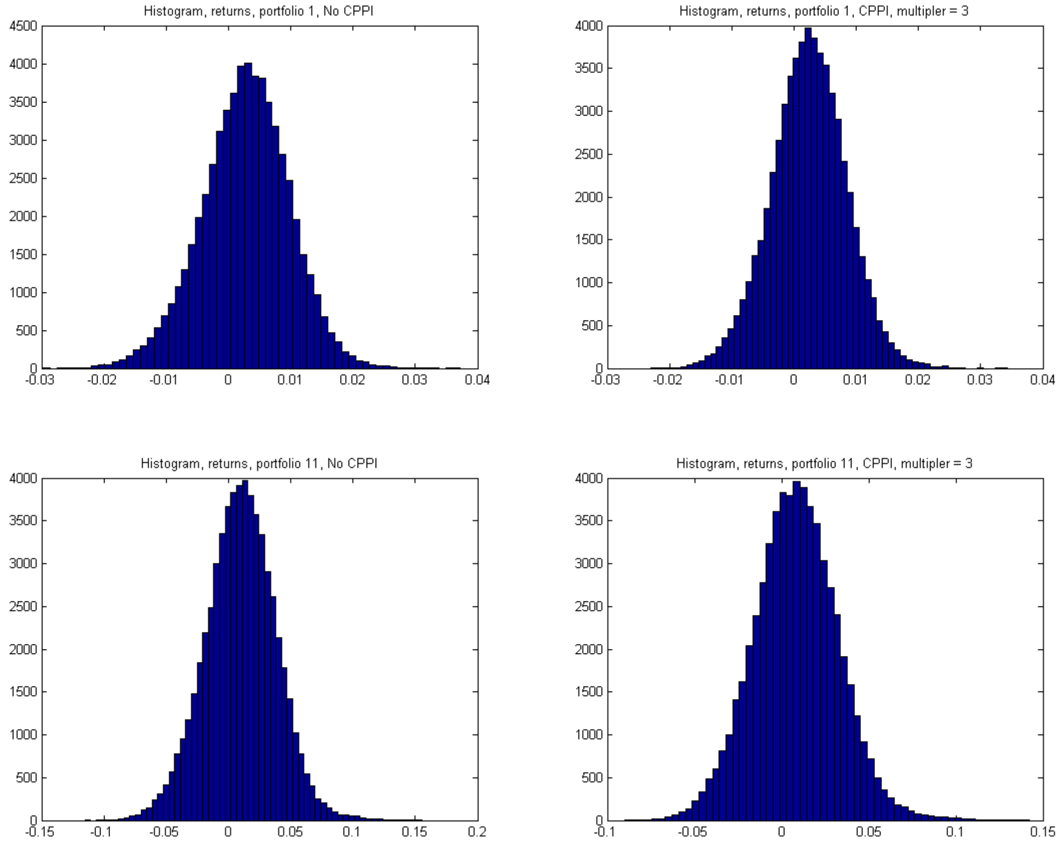


Figure 7.3.1: Histograms of portfolio returns without CPPI (left) and with CPPI (right)

We see the expected behavior of the returns and their standard deviation in respect to changes in the multiplier, i.e. a higher multiplier leads to a higher standard deviation and expected return (p. 211, Balder, 2009). We may also note that when the multiplier is eight (which is a quite unreasonably large multiplier) the expected return actually increases relative to a non-CPPI portfolio, however so does also the standard deviation. This may be a result of the CPPI structure allocating more to the risky asset resulting in a slightly higher return, a higher standard deviation and a higher probability of exceeding the gap. However, when using a more reasonable multiplier of 3 we get the results congruent with those of a constrained CPPI (p 1108, Boulier Kanniganti 1995). These results are also congruent with those of Clark and Arnott for portfolios where, as in our case, only a part of the portfolio is insured using CPPI (p 41, Clark, Arnott, 1987).

We can also see that portfolio four that has the smallest proportion allocated to the CPPI also has the smallest change in the mean return and the smallest change in the standard deviation (using the figures with the more reasonable multiplier three).

The histograms above shows, albeit minuscule, changes in the distributions of returns. The distributions changes from a normal distribution-like distribution to more of a log-normal appearance, where we get fewer worst cases but we also lose the greatest (positive) returns as well. The reasoning for only showing the histograms for portfolio one and eleven is that portfolio eleven has the largest proportion of its portfolio allocated to the CPPI and hence portrays the largest change in distribution where as the first portfolio has a CPPI allocation more representative of the majority of portfolios.

In table 7.3.3 we see a quite large change in the 99.5-quantiles when using CPPI, at least in the light of a smaller proportion of each portfolio being allocated to the CPPI structure and in table 7.3.1 we see how negative returns portray the strengths of CPPI in how smaller the returns become when not using CPPI.

We can therefore conclude that in the case were a management actions structure, such as CPPI, is in use it may be prudent to examine its implications on the risk measures as its implications might be large. However one must also keep in mind that the extension of a risk model to incorporate management actions may increase the models complexity and perhaps more importantly, the number of assumptions. As in this particular case where we needed to assume a gap risk for one month periods, i.e. it is may be hard to connect the actual one day-multiplier used in capital management to the one used in the risk model.

8. Conclusions

Regarding scaling of the 99% Value-at-Risk to time horizons of periods of ten days up to one year we find that the square-root-of-time rule performs well compared to the other methods investigated in this study. In the case of quarterly and yearly VaR for the fixed income indices used here it has satisfactory performance for the less volatile assets, Swedish government bonds (OMRX-Bond Index) and European investment grade bonds (FTSE Euro Corp. Bond Index). It also performs well for emerging market bonds (JPMorgan Em. Mark. Bond Index). The sharp drops in price levels in late 2008 for American corporate bonds (Dow Jones Corp. Bond Index) and high yielding bonds (Credit Suisse High Yield Index) combined with short price history makes scaling VaR for these assets difficult.

Considering the ease of implementation and the conservative nature of the square-root-of-time rule it is appropriate for practical implementation. This implementation should be carried out in the two-step method described here for longer time horizons, where one first estimate an intermediate VaR, e.g. one month, and then us the scaling procedure to obtain the VaR of choice.

While implementations of management action models are attractive in the way they may give a more accurate picture of the world they try to model, one must also keep in mind that they can increase the complexity and number of assumptions in the model. However, CPPI is a fairly simple and stringent model quite suitable for implementations in the light of regulators allowing management actions in internal risk models.

An interesting area of future research is the effects of CPPI implemented on a more realistic daily return basis on portfolio risk, as well as considering other management actions.

9. Bibliography

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