KTH Mathematics

Examination in SF2974 Portfolio Theory and Risk Management, October 18, 2010, 14:00–19:00.

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Allowed technical aids: calculator.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

Unless stated otherwise you may assume that it is possible to take positions corresponding to fractions of assets.

GOOD LUCK!

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Problem 1

Consider an investor who is an expected utility maximizer with a power utility function $u(x) = x^\beta$, β in $(0, 1)$. The investor has 100 Euro and has the opportunity to either take a long position in a defaultable bond or deposit the 100 Euro on a risk-free bank account that pays no interest. One bond costs 97 Euro now and pays 100 Euro six months from now if the issuer does not default and 0 in case the issuer defaults. The investor believes that the default probability is 0.02 and decides not to buy the bond. What can be said about β? (10 p)

Problem 2

Consider another investor who is an expected utility maximizer with a power utility function $u(x) = \sqrt{x}$. The investor has 100 Euro and has the opportunity to take long positions in a defaultable bond and a credit default swap on this bond. One bond costs 97 Euro now and pays 100 Euro six months from now if the issuer does not default and 0 in case the issuer defaults. The credit default swap costs 4 Euro and pays 100 Euro six months from now if the bond issuer defaults and nothing otherwise. The investor believes that the default probability is 0.02. How much of the 100 Euro does the investor invest in the bond? How much in the credit default swap? (10 p)

Problem 3

Consider yet another investor that has the same investment opportunities as the investor in Problem 2 and also consider the default probability to be 0.02. Let V be the value in Euro of the investor's position at the maturity of the bond.

(a) Suppose the investor has 100 Euro and wants to maximize the expected value of V subject to the risk constraint $\mathrm{VaR}_{0.05}(V - 100) \leq 10$ (and the budget constraint and only long positions). How much of the 100 Euro does the investor invest in the bond? How much in the credit default swap? (5 p)

(b) Suppose now that the risk constraint is replaced by $\mathrm{ES}_{0.05}(V - 100) \leq 10$. What is the effect of the new risk constraint on the solution to the investment problem? (5 p)

Problem 4

Consider an investor who may invest in two risky assets and a risk-free zero-coupon bond maturing one year from now. The risky assets do not pay any dividends and both long and short positions in the three assets are allowed. The investor has the initial capital 1000 Euro and may invest some or all of it. The investor wants to maximize the expected value of her wealth one year from now. However, she is only allowed to take positions whose value one year from now has a standard deviation of at most 300. The investor believes that the percentage returns of the risky assets have the expected values 1.1 and 1.2, that the standard deviations of the percentage returns are 0.3 and 0.5 (in that order), and that the linear correlation coefficient between the two percentage returns is 0.5. The percentage return for the risk-free bond is 1. How much money is invested in each of the three assets? (10 p)

Problem 5

A bank has issued a European call option with strike price 110 and maturity in one year from now on a stock market index that pays no dividends. The bank wants to hedge the issued contract by taking positions in a risk-free one-year zero-coupon bond with face value 1 and in the underlying index. According to the bank, the logarithm (the natural logarithm log) of the index value one year from now is normally distributed with mean log(100) and standard deviation 0.3. The bank wants the value A of the hedge one year from now to match the value L of the call option payoff as well as possible in the sense that the expectation of the squared difference between the two should be minimized. The bond costs 0.97 now and one share of the index costs 100 now. The covariance between the value of the index and the payoff of the option at maturity is approximately 570.16 and the expected option payoff is approximately 10.27, according to the view of the bank. Determine the position in the bond and the index that gives the optimal hedge of the call option. It is allowed here to take long and short positions. Also plot the value of $A - L$ as a function of the value of the index one year from now. (10 p)

Problem 1

$\mathrm{E}[u(V)] = (1-p)(100V_0/B_0)^\beta = 0.98(100^2/97)^\beta$ and $u(V_0) = 100^\beta$. No investment in the bond means that

$$0.98(100^2/97)^\beta \leq 100^\beta \Leftrightarrow 0.98^{1/\beta} \leq 0.97 \Leftrightarrow \frac{1}{\beta}\log(0.98) \leq \log(0.97)$$
$$\Leftrightarrow \beta\log(0.97) \geq \log(0.98) \Leftrightarrow \beta \leq \frac{\log(0.98)}{\log(0.97)} \approx 0.663271$$

Problem 2

The solution to

$$\begin{array}{ll} \text{maximize} & \mathrm{E}[(w_1c_1^{-1}X_1 + w_2c_2^{-1}X_2)^\beta] \\ \text{subject to} & w_1 + w_2 \leq V_0 \\ & w_1 \geq 0, w_2 \geq 0, \end{array}$$

where $X_k \sim Be(p_k)$ and $X_1 + X_2 = 1$, is

$$w_k = V_0\Big(\frac{c_k^\beta}{\beta p_k}\Big)^{1/(\beta-1)} \Big/ \Big\{\Big(\frac{c_1^\beta}{\beta p_1}\Big)^{1/(\beta-1)} + \Big(\frac{c_2^\beta}{\beta p_2}\Big)^{1/(\beta-1)}\Big\}$$

Here $c_1 = 0.97$, $c_2 = 0.04$, $p_1 = 0.98$, $p_2 = 0.02$, $V_0 = 100$, $\beta = 0.5$. This gives $(w_1, w_2) \approx (99, 1)$.

Problem 3

We have $V = w_1c_1^{-1}100I + w_2c_2^{-1}100(1-I)$, where $c_1 = 97$, $c_2 = 4$, and I is an indicator variable with $\mathrm{P}(I = 1) = 0.98$. Therefore

$$\mathrm{VaR}_p(V - 100) = 100 + \mathrm{VaR}_p(100w_2c_2^{-1} + 100(w_1c_1^{-1} - w_2c_2^{-1})I)$$

which gives

$$\begin{aligned}\mathrm{VaR}_p(V - 100) =& 100 - 100w_2c_2^{-1} \\ &+ \begin{cases} 100(w_1c_1^{-1} - w_2c_2^{-1})\,\mathrm{VaR}_p(I) & \text{if } w_1c_1^{-1} \geq w_2c_2^{-1}, \\ 100(w_2c_2^{-1} - w_1c_1^{-1})\,\mathrm{VaR}_p(-I) & \text{if } w_1c_1^{-1} < w_2c_2^{-1}. \end{cases}\end{aligned}$$

We have $\mathrm{VaR}_p(I) = F_{-I}^{-1}(1-p)$ and $\mathrm{VaR}_p(-I) = F_I^{-1}(1-p)$ and

$$F_{-I}^{-1}(1-p) = \begin{cases} -1 & \text{if } 1-p \in (0, 0.98], \\ 0 & \text{if } 1-p \in (0.98, 1), \end{cases}$$

and

$$F_I^{-1}(1-p) = \begin{cases} 0 & \text{if } 1-p \in (0, 0.02], \\ 1 & \text{if } 1-p \in (0.02, 1). \end{cases}$$

This gives

$$\mathrm{VaR}_p(V - 100) = 100 - \begin{cases} 100w_1c_1^{-1} & \text{if } p \in (0.02, 0.05], \\ 100\min(w_1c_1^{-1}, w_2c_2^{-1}) & \text{if } p \in (0, 0.02]. \end{cases}$$

In particular, $\mathrm{VaR}_{0.05}(V-100) = 100-100w_1c_1^{-1}$ and therefore $\mathrm{VaR}_{0.05}(V-100) \leq 10$ is equivalent to $w_1 \geq 87.3$. Therefore the solution to the optimization problem with the VaR-constraint is $(w_1, w_2) = (100, 0)$.
Now that $\mathrm{VaR}_p(V - 100)$ has been computed for $p \leq 0.05$ it is easy to compute $\mathrm{ES}_{0.05}(V - 100)$:

$$\begin{aligned}\mathrm{ES}_{0.05}(V-100) &= \frac{1}{0.05}\int_0^{0.05} \mathrm{VaR}_p(V-100)dp \\ &= \begin{cases} 100 - 100w_1c_1^{-1} & \text{if } w_1c_1^{-1} < w_2c_2^{-1}, \\ 100 - 100\frac{3}{5}w_1c_1^{-1} - 100\frac{2}{5}w_2c_2^{-1} & \text{if } w_1c_1^{-1} \geq w_2c_2^{-1}. \end{cases}\end{aligned}$$

With $w_2 = 100 - w_1$ we find that $w_1c_1^{-1} < w_2c_2^{-1}$ is equivalent to $w_1 < 96.0396$. We want w_1 as large as possible and therefore consider the case $w_1 \geq 96.0396$. In this case, $\mathrm{ES}_{0.05}(V-100)$ together with $w_2 = 100-w_1$ is equivalent to $w_1 \leq 97$. Therefore the solution to the optimization problem with the ES-constraint is $(w_1, w_2) = (97, 3)$.

Problem 4

The solution to

$$\begin{array}{ll} \text{maximize} & w_0 + \boldsymbol{\mu}^{\mathrm{T}}\mathbf{w} \\ \text{subject to} & \mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w} \leq 300^2 \\ & w_0 + \mathbf{1}^{\mathrm{T}}\mathbf{w} \leq 1000 \end{array}$$

is (see LN)

$$\mathbf{w} = 300\frac{\Sigma^{-1}(\boldsymbol{\mu} - \mathbf{1})}{\sqrt{(\boldsymbol{\mu} - \mathbf{1})^{\mathrm{T}}\Sigma^{-1}(\boldsymbol{\mu} - \mathbf{1})}}, \quad w_0 = 1000 - \mathbf{1}^{\mathrm{T}}\mathbf{w}.$$

The solution here is $(w_0, w_1, w_2) \approx (149.70, 414.78, 435.52)$.

Problem 5

We look for (h_0, h_1) that minimizes $\mathrm{E}[(h_0 + h_1S - \max(S - 110, 0))^2]$. The solution is $h_1 = \mathrm{Cov}(\max(S - 110, 0), S)/\operatorname{Var}(S)$ and $h_0 = \mathrm{E}[\max(S - 110, 0)] - h_1\,\mathrm{E}[S]$. Since $\mathrm{E}[S] = e^{\log(100)+0.3^2/2} \approx 104.60$ and $\mathrm{E}[S^2] = \mathrm{E}[e^{2\log(100)+2\cdot 0.3Z}] = e^{2\log(100)+2\cdot 0.3^2}$ we find that $\operatorname{Var}(S) \approx 1030.43$. This gives $(h_0, h_1) \approx (-47.61, 0.55)$.