EXAMINATION IN SF2975 FINANCIAL DERIVATIVES

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Suggested solutions

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Problem 1

(a) We use Feynman-Kac. It follows that

$$F(t,x) = E_{t,x}\left[e^{a(T-t)}X(T)\right],$$

where X has dynamics

$$dX(t) = bX(t)dW(t),$$

and W is a 1-dimensional Wiener process. The solution to this SDE is

$$X(T) = X(t)e^{-\frac{b^2}{2}(T-t)+b(W(T)-W(t))},$$

and we get

$$\begin{aligned} F(t,x) &= e^{a(T-t)}E_{t,x}\left[xe^{-\frac{b^2}{2}(T-t)+b(W(T)-W(t))}\right] \\ &= e^{a(T-t)}x\underbrace{E\left[e^{-\frac{b^2}{2}(T-t)+b(W(T)-W(t))}\right]}_{=1} \\ &= e^{a(T-t)}x. \end{aligned}$$

Alternatively we realise that X is a martingale, from which it follows that

$$F(t,x) = e^{a(T-t)}E_{t,x}[X(T)] = e^{a(T-t)}x.$$

(b) We know that if $f : [0,T] \to \mathbb{R}$ satisfies

$$\int_0^T f^2(u)du < \infty,$$

then

$$\int_0^T f(u)dW(u) \sim N\left(0, \sqrt{\int_0^T f^2(u)du}\right).$$

In our case $f(u) = u^2$. Since

$$\int_0^T \left(u^2\right)^2 du = \int_0^T u^4 du = \frac{T^5}{5} < \infty,$$

we get

$$X = \int_0^T u^2 dW(u) \sim N\left(0, \sqrt{\frac{T^5}{5}}\right).$$

(c) See the book.

Problem 2

(a) The risk-neutral valuation formula yields

$$\Pi(t;X) = e^{-r(T-t)} E^Q \left[(\ln(S(T)))^2 \Big| \mathcal{F}_t \right],$$

where under Q the dynamics of S are given by

$$dS(t) = rS(t)dt + \sigma S(t) dW^Q(t).$$

Since

$$S(T) = S(t) e^{\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W^Q(T) - W^Q(T))}$$

we get

$$\ln S(T) = \ln S(t) + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W^Q(T) - W^Q(T)).$$

It follows that

$$\ln S(T) | \mathcal{F}_t \sim N\left(\ln S(t) + (r - \sigma^2/2)(T-t), \sigma\sqrt{T-t}\right).$$

Now

$$\begin{aligned} E^Q \left[(\ln S(T))^2 \Big| \mathcal{F}_t \right] &= \mathrm{Var}^Q (\ln S(T) | \mathcal{F}_t) + \left(E^Q [\ln S(T) | \mathcal{F}_t]\right)^2 \\ &= \sigma^2 (T-t) + \left(\ln S(t) + (r - \sigma^2/2)(T-t)\right)^2, \end{aligned}$$

and for $t \in [0, T]$

$$\Pi(t;X) = e^{-r(T-t)} \left(\sigma^2(T-t) + \left(\ln S(t) + (r - \sigma^2/2)(T-t)\right)^2\right).$$

(b) We know that the hedging portfolio is given by

$$h^S(t) = \frac{\partial F}{\partial s}(t, S(t)),$$

where

$$\begin{aligned} F(t,s) &= E^Q_{t,s}\left[(\ln S(T))^2\right] \\ &= e^{-r(T-t)} \left(\sigma^2(T-t) + \left(\ln s + (r - \sigma^2/2)(T-t)\right)^2\right), \end{aligned}$$

and that

$$h^B(t) = \frac{F(t, S(t)) - S(t)\frac{\partial F}{\partial s}(t, S(t))}{B(t)}.$$

It follows that

$$h^S(t) = \frac{2}{S(t)} e^{-r(T-t)} \left(\ln S(t) + (r - \sigma^2/2)(T-t)\right)$$

and

$$h_B(t) = e^{-rT} \left[2(\sigma^2 - r)(T-t) + \left(\ln S(t) + (r-\sigma^2/2)(T-t)\right)^2 - 2\ln S(t)\right].$$

Problem 3

(a) The Q-dynamics of $f(t,T)$ are given by

$$df(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du\, dt + \sigma(t,T)dW^Q(t),$$

where W^Q is a 1-dimensional Q-Wiener process. With

$$\sigma(t,T) = \frac{\sigma_0}{1+a(T-t)}$$

we get the following drift under Q:

$$\begin{aligned}
\sigma(t,T)\int_t^T \sigma(t,u)du &= \frac{\sigma_0}{1+a(T-t)}\int_t^T \frac{\sigma_0}{1+a(u-t)}du \\
&= \frac{\sigma_0}{1+a(T-t)}\left[\frac{\sigma_0}{a}\cdot \ln(1+a(u-t))\right]_t^T \\
&= \frac{\sigma_0^2}{a}\cdot\frac{\ln(1+a(T-t))}{1+a(T-t)}.
\end{aligned}$$

Hence

$$df(t,T) = \frac{\sigma_0^2}{a}\cdot\frac{\ln(1+a(T-t))}{1+a(T-t)}dt + \frac{\sigma_0}{1+a(T-t)}dW^Q(t).$$

The initial value is the the same as under P:

$$f(0,T) = f^\star(0,T).$$

Under Q, the dynamics of $p(t,T)$ is given by

$$dp(t,T) = r(t)p(t,T)dt + \left(-\int_t^T \sigma(t,u)du\right)p(t,T)dW^Q(u).$$

In our case

$$-\int_t^T \sigma(t,u)du = -\int_t^T \frac{\sigma_0}{1+a(u-t)}du = -\frac{\sigma_0}{a}\ln(1+a(T-t)),$$

so we get Q-dynamics

$$dp(t,T) = r(t)p(t,T)dt - \frac{\sigma_0}{a}\ln(1+a(T-t))p(t,T)dW^Q(t)$$

with initial values

$$p(0,T) = e^{-\int_0^T f^\star(0,u)du}.$$

(b) See the book.

Problem 4

(a) We know that for $t \in [0, T]$

$$\Pi(t; X) = p(t,T) E^{Q^T}\left[X | \mathcal{F}_t\right],$$

where $p(t,T)$ is the price at time t of a ZCB maturing at T and Q^T is the T-forward measure. The model

$$dr(t) = \sigma_0 dW^Q(t)$$

has an ATS with, in the language of the course book,

$$\alpha(t) = \beta(t) = \gamma(t) = 0 \text{ and } \delta(t) = \sigma_0^2.$$

Hence, the ZCB prices in this model are given by $p(t,T) = e^{A(t,T) - B(t,T)r(t)}$, where A and B solves the system of equations

$$\begin{cases} \frac{\partial A(t,T)}{\partial t} &= -\frac{\sigma_0^2}{2} B^2(t,T) \\ A(T,T) &= 0 \end{cases}$$

and

$$\begin{cases} \frac{\partial B(t,T)}{\partial t} &= -1. \\ B(T,T) &= 0 \end{cases}$$

The solution to the last equation is given by

$$B(t,T) = T - t,$$

and we then get

$$A(T,T) - A(t,T) = -\int_t^T \frac{\sigma_0^2}{2}(T-u)^2 du = -\frac{\sigma_0^2}{2} \cdot \frac{(T-t)^3}{3},$$

or

$$A(t,T) = \frac{\sigma_0^2 (T-t)^3}{6}.$$

Hence

$$p(t,T) = e^{\frac{\sigma_0^2 (T-t)^3}{6} - (T-t)r(t)}.$$

The forward rates are given by

$$f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T} = \frac{\partial}{\partial T}\left((T-t)r(t) - \frac{\sigma_0^2 (T-t)^3}{6}\right) = r(t) - \frac{\sigma_0^2 (T-t)^2}{2}.$$

(b) In general

$$\begin{aligned} dr(t) &= \mu(t, r(t))dt + \sigma(t, r(t))dW(t) \\ &= \mu_Q(t, r(t))dt + \sigma(t, r(t))dW^Q(t), \end{aligned}$$

where W and W^Q is a Wiener process under P and Q respectively. We know that absence of arbitrage implies

$$\mu_Q(t, r(t)) = \mu(t, r(t)) - \lambda(t, r(t))\sigma(t, r(t)).$$

In our case we have

$$\mu_Q(t, r(t)) = 0,\ \lambda(t, r(t)) = \lambda_0 \text{ and } \sigma(t, r(t)) = \sigma_0,$$

which implies

$$\mu(t, r(t)) = \lambda_0\sigma_0$$

and

$$dr(t) = \lambda_0\sigma_0 dt + \sigma_0 dW(t).$$

Thus

$$r(T) = r(0) + \lambda_0\sigma_0 T + \sigma_0 W(T) \sim N\left(r(0) + \lambda_0\sigma_0 T, \sigma_0\sqrt{T}\right),$$

and we get

$$\begin{aligned} P(r(T) < 0) &= P(r(T) \leq 0) \\ &= P\left(\frac{r(T) - r(0) - \lambda_0\sigma_0 T}{\sigma_0\sqrt{T}} \leq \frac{-r(0) - \lambda_0\sigma_0 T}{\sigma_0\sqrt{T}}\right) \\ &= \Phi\left(-\frac{r(0) + \lambda_0\sigma_0 T}{\sigma_0\sqrt{T}}\right). \end{aligned}$$

Problem 5

(a) We know that $S_f(t)X(t)/B_d(t)$ should be a Q^d-martingale. First of all

$$\begin{aligned} d(S_f(t)X(t)) &= S_f(t)dX(t) + X(t)dS_f(t) + dS_f(t)dX(t) \\ &= \alpha_X S_f(t)X(t)dt + \sigma_X S_f(t)X(t)dW(t) + \alpha_f S_f(t)X(t)dt \\ &\quad +\sigma_f S_f(t)X(t)dW(t) + \sigma_f\sigma_X S_f(t)X(t)dt \\ &= (\alpha_X + \alpha_f + \sigma_f\sigma_X)\frac{S_f(t)X(t)}{B_d(t)}dt + (\sigma_X + \sigma_f)S_f(t)X(t)dW(t). \end{aligned}$$

We then get

$$
\begin{aligned}
d\left(\frac{S_f(t)X(t)}{B_d(t)}\right) &= d\left(e^{-r_d t}[S_f(t)X(t)]\right) \\
&= (\alpha_X + \alpha_f + \sigma_f\sigma_X - r_d)\frac{S_f(t)X(t)}{B_d(t)}dt \\
&\quad +(\sigma_X + \sigma_f)\frac{S_f(t)X(t)}{B_d(t)}dW(t) \\
&= (\alpha_X + \alpha_f + \sigma_f\sigma_X - r_d)\frac{S_f(t)X(t)}{B_d(t)}dt \\
&\quad +(\sigma_X + \sigma_f)\frac{S_f(t)X(t)}{B_d(t)}\left(dW^d(t) + \varphi(t)dt\right) \\
&= (\alpha_X + \alpha_f + \sigma_f\sigma_X - r_d + (\sigma_X + \sigma_f)\varphi(t))\frac{S_f(t)X(t)}{B_d(t)}dt \\
&\quad +(\sigma_X + \sigma_f)\frac{S_f(t)X(t)}{B_d(t)}dW^d(t),
\end{aligned}
$$

where W^d is a Q^d-Wiener process. By choosing

$$\varphi(t) = -\frac{\alpha_X + \alpha_f + \sigma_f\sigma_X - r_d}{\sigma_X + \sigma_f}$$

we see that the discounted price process is a Q^d-martingale. It follows that the dynamics of X under Q^d is

$$
\begin{aligned}
dX(t) &= \alpha_X X(t)dt + \sigma_X X(t)\left(dW^d(t) - \frac{\alpha_X + \alpha_f + \sigma_f\sigma_X - r_d}{\sigma_X + \sigma_f}\right) \\
&= \left(\alpha_X - \sigma_X\frac{\alpha_X + \alpha_f + \sigma_f\sigma_X - r_d}{\sigma_X + \sigma_f}\right)X(t)dt + \sigma_X X(t)dW^d(t).
\end{aligned}
$$

(b) The price at $t \in [0,T]$ of the T-claim $Z = S_f(T)X(T)$ is given by

$$
\begin{aligned}
\Pi(t;Z) &= e^{-r_d(T-t)}E^{Q^d}\left[S_f(T)X(T)|\mathcal{F}_t\right] \\
&= e^{r_d t}E^{Q^d}\left[e^{-r_d T}S_f(T)X(T)|\mathcal{F}_t\right] \\
&= \left\{e^{-r_d t}S_f(t)X(t) \text{ is a } Q^d\text{-martingale}\right\} \\
&= e^{r_d t}e^{-r_d t}S_f(t)X(t) \\
&= S_f(t)X(t).
\end{aligned}
$$