Numerical investigations of the distributions of the longest excursions in tied down simple Random Walks and Brownian Bridges

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Abstract

This paper studies the distributions of the longest excursion in tied down simple random walks and the longest excursions in the Brownian bridge. New results are presented, numerically investigated and related to old results. The distributions for the longest excursions in the Brownian bridge are apprehended by passing to the limit in the random walk.

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Contents

0 Summary

We are studying the distributions of the longest excursion in tied down simple random walks and the longest excursions in the Brownian bridge. The distributions for the excursions in the Brownian bridge are apprehended through passing to the limit in the random walk.

Two different representations for the longest excursion in a tied down simple random walk are presented. One representation is apprehended using combinatorial methods and the other by using generating functions. The first representation gives a recursive relation for the distribution and the second representation gives an explicit formula. Similar results are presented in Wendel:[11] and Holst:[8].

The density of the longest excursion in the Brownian bridge is obtained in Pitman and Yor:[9] for the interval $1/4 < x < 1$. The limit of the recursive formula for the random walk gives a recursive formula that replicates the density results for the interval $1/3 < x < 1$. This interval is also covered by Gourdon:[7].

Finally, a Kummer representation, due to Holst:[8], is presented, that gives expressions for the distribution of the n:th longest excursion. Explicit formulas for the three longest excursions are given. These formulas are represented as infinite sums involving the zeros of a certain Kummer function. Fortunately, a truncation at the term involving only the first zero gives accurate approximations. The formula for the longest excursion, due to Rosén, is an old result and is present in Wendel:[11].

1 Introduction

1.1 Random walk–notation

Let $\{\xi_i\}_{i=1}^{\infty}$ be independent random variables with

$$\mathbf{P}(\xi_i = -1) = \mathbf{P}(\xi_i = +1) = \frac{1}{2},$$

and define $\{S_n\}_{n=1}^{\infty}$ by $S_n = \sum_{i=1}^{n} \xi_i$. This is often referred to as a simple symmetric random walk. We shall use a geometric terminology and refer to rectangular coordinates t,x; for definiteness we imagine the t-axis is horizontal, the x-axis vertical. The arrangement $(\xi_1, \ldots, \xi_n)$ will be presented by a polygonal line whose k:th side has slope ξ_k and k:th vertex has ordinate S_k. Such lines will be called paths. This structure is standard. A nice presentation is given in [5]: Feller I (1968), from which some of the examples to come are taken.

Definition 1.1.1 *Let $n > 0$ and x be integers. A path $(S_1, \ldots, S_n)$ from the origin to the point (n,x) is a polygonal line whose vertices have abscissas $0, 1, \ldots, n$ and ordinates $S_0, S_1, \ldots, S_n$ satisfying the above with $S_n = x$. We shall refer to n as the length of the path.*

To calculate probabilities of events in finite time, we can simply count the paths that matches the event and divide by the total number of possible paths. This is then a combinatorial problem.

Unless otherwise mentioned, we will think of a random walk as starting in $S_0 = 0$.

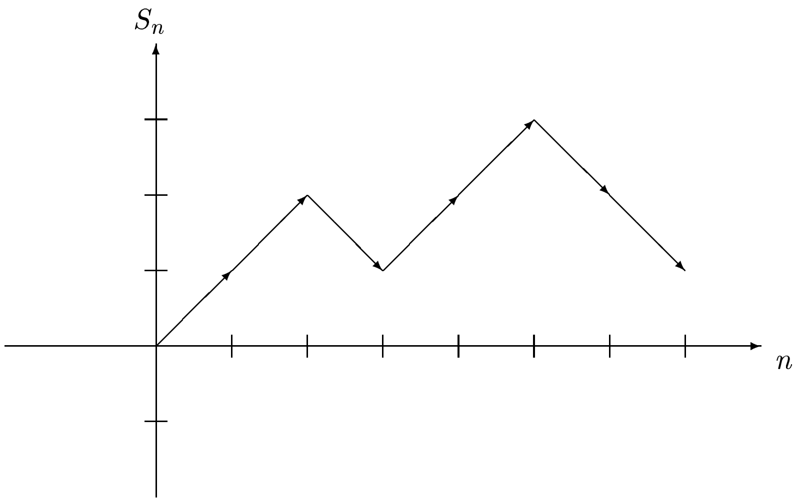

Figure 1: Realisation of a simple symmetric random walk.

1.2 Tied down random walk and excursions

This paper is about excursions in tied down random walks and Brownian bridges.

Definition 1.2.1 *We say that a random walk is tied down if we condition on the event $\{S_n = 0\}$. n is then bound to be an even number (if the random walk starts at $\{S_0 = 0\}$) and we usually write S_{2n} to emphasize this and to simplify calculations.*

Definition 1.2.2 *An excursion in a random walk is the number of steps between two successive returns to the origin.*

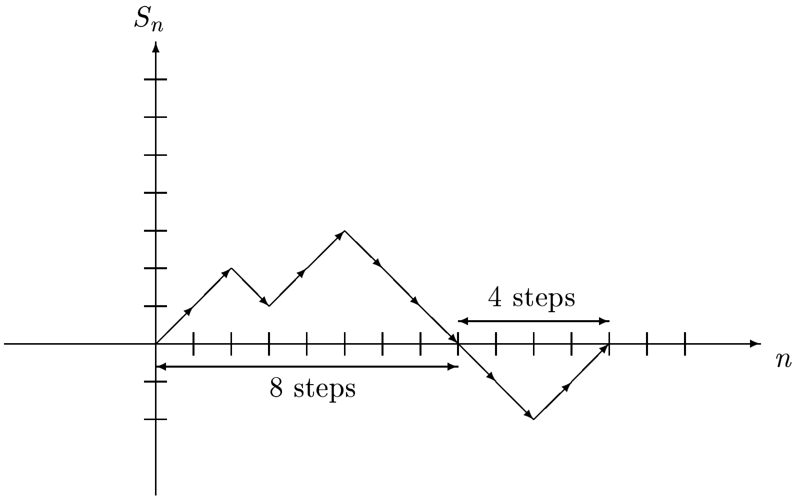

Figure 2: Example of two excursions in a tied down random walk.

1.3 Brownian motion, Brownian bridge and excursions in a Brownian motion

It is a well known fact that the continuous analogue of a simple symmetric random walk is the Brownian motion. What is meant by this statement and other related issues is found in [3]: Billingsley (1968). The idea is to scale the time and position of the random walk in a suitable way, and then passing to the limit. The invariance principle states that any random walk, with variance 1, has this asymptotic behaviour. In particular, we can choose to look at the simple random walk. Donsker's theorem (or the functional central limit theorem) then makes it possible to derive results relating to the Brownian motion by doing the calculations for the simple symmetric random walk and passing to the limit. We 'only' need to show that the functional we are interested in (for example the longest excursion) is continuous. More on this is in Section 6.

Recall the following definitions.

Definition 1.3.1 *Let* $\{B_t\}$ *be a standard Brownian motion starting at zero. The stochastic process* $\{B_t^*\}$ *defined by*

$$B_t^* := B_t - tB_1,\ 0 \leq t \leq 1,$$

is called the Brownian bridge.

Definition 1.3.2 *An excursion in a Brownian motion is the elapsed time between two successive returns to the* t*-axis.*

1.4 Outline of the paper

Section 2 gives some preliminaries on random walks and states some results that are needed later in the paper. The distribution of the longest excursion in a tied down random walk is considered in Section 3. Two different approaches are explored. Section 4 takes the analysis in Section 3 to the limit and evaluates the distribution of the longest excursion in the Brownian bridge. Different representations are studied, for which a Kummer representation turns out to be the most general one. The different representations are stated and compared to each other and also to the discrete random walk. A general result of the Kummer representation also makes it possible to get the distribution of the n:th longest excursion, which is studied in Section 5. Issues of weak convergence of functionals, such as the functional that extracts the longest excursion, are discussed in Section 6. Conclusions are given in Section 7. Some other issues that are not in the text are given in the appendices.

2 Preliminaries on Random Walks

In this section, we will give an introduction to how calculations for random walks may be done. We will derive results that are needed later, when we study excursions in random walks. The results in this section can be found in [5].

2.1 A first example of a simple calculation

Example 2.1.1 *Calculate the probability* $\mathbf{P}(S_n = x) = p_{n,x}$.

We shall do this by counting sample paths. There are 2^n paths of length n. The number x is composed by a $+1$:s and b -1:s, i.e.

$$x = a - b, \quad n = a + b.$$

In this case the a places for the positive ξ_k can be chosen from the $n = a+b$ places in

$$N_{n,x} = \binom{a+b}{a}$$

different ways.

We can then calculate the desired probability as

$$\mathbf{P}(S_n = x) = p_{n,x} = \binom{a+b}{a} \cdot \left(\frac{1}{2}\right)^n = \binom{n}{\frac{n+x}{2}} \cdot \left(\frac{1}{2}\right)^n.$$

□

Especially we have

$$\mathbf{P}(S_{2n} = 0) = u_{2n} = \binom{2n}{n} \cdot \left(\frac{1}{2}\right)^{2n}. \tag{1}$$

2.2 The reflection principle and the Ballot theorem

Let A be the coordinate (a, α) and B be the coordinate (b, β), where $b > a \geq 0$ and $\alpha > 0$, $\beta > 0$. By reflection of A on the t-axis is meant $\tilde{A} = (a, -\alpha)$. A path from A to B is defined in the obvious manner.

Lemma 2.2.1 *(Reflection principle) The number of paths from A to B which touch or cross the t-axis equals the number of all paths from $\tilde{A}$ to B.*

For a proof, see [5].

Theorem 2.2.1 *(The Ballot theorem) Let n and x be positive integers. There are exactly $\frac{x}{n}N_{n,x}$ paths from the origin to the point (n, x) such that $S_1 > 0, \ldots, S_n > 0$.*

Proof. Clearly there exists exactly as many admissible paths as there are paths from the point $(1, 1)$ to (n, x) which neither touch or cross the t-axis. By Lemma 2.2.1 the number of such paths equal

$$N_{n-1,x-1} - N_{n-1,x+1} = \binom{a+b-1}{a-1} - \binom{a+b-1}{a} = \frac{x}{n}N_{n,x},$$

where a and b are defined in Section 2.1. □

2.3 An important lemma

It turns out later that the probability u_{2n}, in (1), appears when calculating probabilities of many other events. The following lemma points out a few of these important relations.

Lemma 2.3.1

$$\begin{aligned} u_{2n} &= \mathbf{P}(S_1 \neq 0, S_2 \neq 0, \ldots, S_{2n} \neq 0) = \mathbf{P}(S_1 \geq 0, S_2 \geq 0, \ldots, S_{2n} \geq 0) \\ &= 2\mathbf{P}(S_1 > 0, S_2 > 0, \ldots, S_{2n} > 0). \end{aligned}$$

Proof. When the event $(S_1 \neq 0, S_2 \neq 0, \ldots, S_{2n} \neq 0)$ occurs either all the S_i are positive, or all are negative. Then it is clear that

$$\mathbf{P}(S_1 \neq 0, S_2 \neq 0, \ldots, S_{2n} \neq 0) = 2\mathbf{P}(S_1 > 0, S_2 > 0, \ldots, S_{2n} > 0).$$

Considering all the possible values of S_{2n} it is clear that

$$\mathbf{P}(S_1 > 0, \ldots, S_{2n} > 0) = \sum_{r=1}^{\infty} \mathbf{P}(S_1 > 0, \ldots, S_{2n-1} > 0, S_{2n} = r), \qquad (2)$$

where all terms with $r > n$ vanish. By the Ballot theorem the number of paths satisfying the condition indicated on the right side equals $N_{2n-1,2r-1} - N_{2n-1,2r+1}$, and so the r:th term of the sum equals

$$\frac{1}{2}(p_{2n-1,2r-1} - p_{2n-1,2r+1}).$$

The negative part of the r:th term cancels against the positive part of the $(r+1)$:th term with the result that the sum in (2) reduces to $\frac{1}{2}(p_{2n-1,1})$. It is easily verified that $p_{2n-1,1} = u_{2n}$.

Left to prove is that $\mathbf{P}(S_1 > 0, S_2 > 0, \ldots, S_{2n} > 0) = \frac{1}{2}\mathbf{P}(S_1 \geq 0, S_2 \geq 0, \ldots, S_{2n} \geq 0)$. Indeed, a path of length $2n$ with all vertices strictly above the t-axis passes through the point (1,1). Taking this point as new origin we obtain a new path of length $2n-1$ with all vertices above or on the new t-axis. It follows that

$$\mathbf{P}(S_1 > 0, S_2 > 0, \ldots, S_{2n} > 0) = \frac{1}{2}\mathbf{P}(S_1 \geq 0, S_2 \geq 0, \ldots, S_{2n-1} \geq 0).$$

But S_{2n-1} is an odd number, and hence $S_{2n-1} \geq 0$ implies that also $S_{2n} \geq 0$. This proves the assertion. □

2.4 r:th return to the origin

Let $\{Y_i\}_{i=1}^{n}$ be i.i.d. and represent the steps between the $(i-1)$:th and i:th return to the origin. Y_i is then called the i:th excursion time. The following relation is found in [5]:

$$f_{2n}^{(r)} \equiv \mathbf{P}(Y_1 + \ldots + Y_r = 2n) = \frac{r}{2n-r}\binom{2n-r}{n}\left(\frac{1}{2}\right)^{2n-r}.$$

The distribution of Y_1, which is the number of steps before the first return to the origin, is then given by

$$f_{2n}^{(1)} = \mathbf{P}(Y_1 = 2n) = \frac{1}{2n-1}\binom{2n}{n}\left(\frac{1}{2}\right)^{2n} = \frac{1}{2n-1}u_{2n}.$$

This result can be derived in the following easy way. We have

$$\{Y_1 = j\} = \{S_1 \neq 0, \ldots, S_{j-1} \neq 0, S_j = 0\}.$$

Assume $n > 0$ and integer. Now, it is easily seen that

$$\mathbf{P}(Y_1 > 2n) = \mathbf{P}(S_1 \neq 0, \ldots, S_{2n} \neq 0) = u_{2n},$$

which gives

$$\begin{aligned} f_{2n}^{(1)} &= \mathbf{P}(Y_1 = 2n) = \mathbf{P}(Y_1 > 2n-2) - \mathbf{P}(Y_1 > 2n) \\ &= u_{2n-2} - u_{2n} = \frac{1}{2n-1}u_{2n}. \end{aligned} \tag{3}$$

2.5 Discrete Arc Sine law

Let R_{2n} =Last visit to origo during first 2n steps.

Theorem 2.5.1 *(Arc Sine Law for last visits.) The probability that up to and including step 2n the last visit to the origin occurs at step 2k is given by*

$$\mathbf{P}(R_{2n} = 2k) = u_{2k}u_{2n-2k}, \quad k = 0, 1, 2, \ldots, n.$$

Proof. We are concerned with paths satisfying the conditions $S_{2k} = 0$ and $S_{2k+1} \neq 0, \ldots, S_{2n} \neq 0$. The first $2k$ vertices can be chosen in $2^{2k}u_{2k}$ different ways. Taking the point $(2k, 0)$ as new origin and using lemma 2.3.1 we see that the next $(2n-2k)$ vertices can be chosen in $2^{2n-2k}u_{2n-2k}$ ways. Dividing by 2^{2n} we get the assertion. □

3 Longest excursion of a tied down random walk

This section explores the characteristics of excursions of tied down simple random walks. We will start with an explicit recursive formula for the distribution of the longest excursion time and end by investigating the behaviour of the longest excursion time using generating functions. Parts of the results in this section are based on [8].

3.1 A recursive relation for the longest excursion

Let $\{S_i\}_{i=0}^{2n}$ be a random walk and condition on the event $\{S_{2n} = 0\}$. Let L_{2n} be the longest excursion length until time $2n$. Set

$$G_{2n}(x) = \mathbf{P}\left(\frac{L_{2n}}{2n} \le x \middle| S_{2n} = 0\right). \tag{4}$$

Then it is obvious that

$$G_{2n}(0) = 0, \quad G_{2n}(1) = 1.$$

We will derive a recursive relation for (4) in a few steps. The first step is to derive an exact expression for the interval $\frac{1}{2} \le x \le 1$. Using this result, it is possible to derive a recursive relation for the whole interval $0 \le x \le 1$.

Lemma 3.1.1

$$G_{2n}(x) = 2 - \frac{u_{2[nx]}}{u_{2n}}, \qquad \frac{1}{2} \le x \le 1,$$

where $[nx]$ *is the integer part of* nx.

Proof. Consider the time interval $n < 2l \le 2n$ and the probability

$$\mathbf{P}(L_{2n} = 2l | S_{2n} = 0)$$

Since $n < L_{2n} \le 2n$, we have

$$\begin{aligned}
\mathbf{P}(L_{2n} = 2l | S_{2n} = 0) &= \frac{\mathbf{P}(L_{2n} = 2l, S_{2n} = 0)}{u_{2n}} \\
&= \frac{1}{u_{2n}} \sum_{k=0}^{n-l} u_{2k} f_{2l}^{(1)} u_{2n-2k-2l} \\
&= \frac{f_{2l}^{(1)}}{u_{2n}} \sum_{k=0}^{n-l} u_{2k} u_{2n-2k-2l} \\
&= \{\text{thm } 2.5.1\} = \frac{f_{2l}^{(1)}}{u_{2n}} \\
&= \{(3)\} = \frac{1}{2l-1} \frac{u_{2l}}{u_{2n}}
\end{aligned}$$

For $\frac{1}{2} \le x \le 1$,

$$\begin{aligned}
G_{2n}(x) &= \mathbf{P}\left(\frac{L_{2n}}{2n} \le x \middle| S_{2n} = 0\right) \\
&= 1 - \mathbf{P}\left(L_{2n} > 2nx \middle| S_{2n} = 0\right) \\
&= \{(5)\} = 1 - \sum_{[nx] < l \le n} \frac{f_{2l}^{(1)}}{u_{2n}} \\
&= 1 - \frac{1}{u_{2n}}\left(\sum_{l > [nx]} f_{2l}^{(1)} - \sum_{l > n} f_{2l}^{(1)}\right) \\
&= \{(3)\} = 1 - \frac{1}{u_{2n}}\left(u_{2[nx]} - u_{2n}\right) = 2 - \frac{u_{2[nx]}}{u_{2n}}
\end{aligned}$$

□

It is possible to get a recursive relation for the probability distribution of L_{2n} for the whole interval $0 \le L_{2n} \le 2n$. This is done in the same way as in the proof of Lemma 3.1.1:

$$\mathbf{P}(L_{2n} = 2l | S_{2n} = 0) = \frac{\mathbf{P}(L_{2n} = 2l, S_{2n} = 0)}{u_{2n}} =$$

$$\frac{1}{u_{2n}} \sum_{j=0}^{n-l} u_{2j} G_{2j}((l-1)/j) f_{2l}^{(1)} u_{2n-2j-2l} G_{2n-2l-2j}(l/(n-l-j)) \quad (5)$$

3.2 The longest excursion using generating functions

We will first derive an expression for $G_{2n}(x)$ using generating functions. Then, we will use this result for numerical calculations to see what happens when we increase the number of steps in the random walk.

3.2.1 A theorem based on generating functions

Some notation is needed. Let

$$\left[\sum_{k=0}^{\infty} a_k z^k\right]_m = \sum_{k=0}^{m} a_k z^k, \qquad [z^n] \sum_{k=0}^{\infty} a_k z^k = a_n.$$

In order to get $G_{2n}(x)$, it is natural to introduce the sequence

$$v_{2n,m} = \mathbf{P}(L_{2n} \le 2m, S_{2n} = 0).$$

Using (3)

$$f_{2k} = \mathbf{P}(S_1 \ne 0, \ldots, S_{2k-1} \ne 0, S_{2k} = 0) = \frac{1}{2k-1} u_{2k},$$

we get

$$v_{2n,m} = \sum_{k=1}^{m} f_{2k} v_{2n-2k,m}, \quad n \geq 1, \qquad v_{0,m} = 1. \tag{6}$$

We will also use the generating function for the sequence $\{f_{2k}\}$

$$F(z) = \sum_{k=1}^{\infty} f_{2k} z^k = 1 - \sqrt{1-z},$$

which can be found in for example [5]. The following result, due to Rosén, is given in [11]: Wendel (1964).

Theorem 3.2.1

$$G_{2n}\left(\frac{2m}{2n}\right) = \frac{[z^n]\frac{1}{[\sqrt{1-z}]_m}}{u_{2n}}.$$

Proof. Consider the generating functions

$$F_m(z) = \sum_{k=1}^{m} f_{2k} z^k = [1 - \sqrt{1-z}]_m, \quad V_m(z) = \sum_{k=0}^{\infty} v_{2k,m} z^k.$$

Relation (6) and the convolution properties of generating functions gives

$$V_m(z) - 1 = V_m(z) F_m(z).$$

Then

$$V_m(z) = \frac{1}{1 - F_m(z)} = \frac{1}{[\sqrt{1-z}]_m},$$

and therefore

$$\mathbf{P}(L_{2n} \leq 2m, S_{2n} = 0) = [z^n]\frac{1}{[\sqrt{1-z}]_m}.$$

Thus

$$G_{2n}\left(\frac{2m}{2n}\right) = \mathbf{P}(L_{2n} \leq 2m | S_{2n} = 0) = \frac{[z^n]\frac{1}{[\sqrt{1-z}]_m}}{u_{2n}}.$$

$\square$

3.2.2 Numerical calculations using Theorem 3.2.1

By using Theorem 3.2.1, it is possible to do exact calculations for different n. We do it, using MATLAB and MAPLE.

First, we study the behaviour when n increases for certain $x = \frac{m}{n}$. Note that the results may be given as exact fractions, but we round to 4 decimals. It seems from table 1 that there is convergence towards some limits. The

$x = m/n$	$n = 4$	$n = 12$	$n = 40$	$n = 100$
$x = 3/4$	.8571	.8493	.8465	.8458
$x = 1/2$	.6286	.6004	.5902	.5876
$x = 1/4$	.2286	.1770	.1582	.1533

Table 1: $G_{2n}(x)$ for different n and x

asymptotics are further investigated in Section 4.

Next, we study the full distribution function $G_{2n}(x)$ for different n and x. We can see in figure 3 that the convergence is faster for larger values of

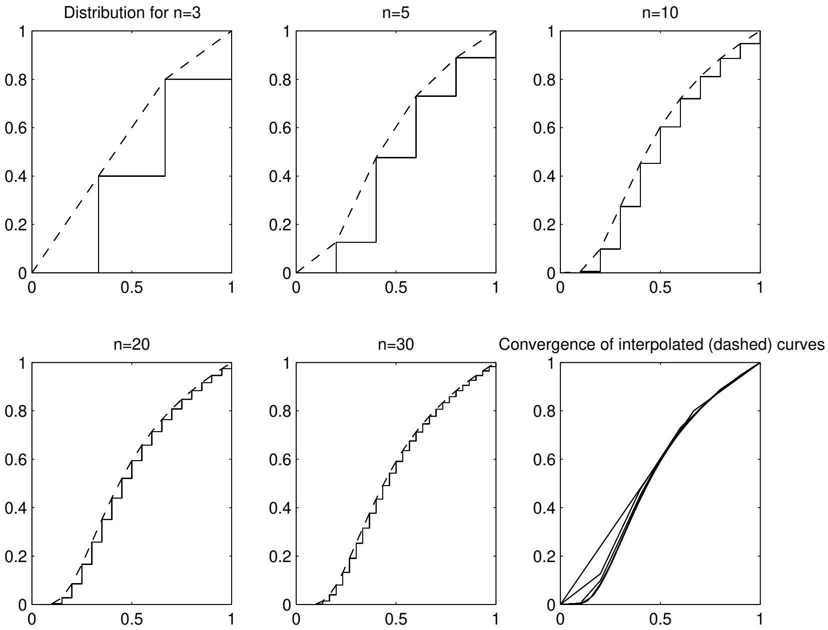

Figure 3: $G_{2n}(x)$ for different n.

x, and the distribution seems to stabilize at such low values as $n = 25$.

Finally, consider the density of the longest excursion. Figure 4 shows the density for different n.

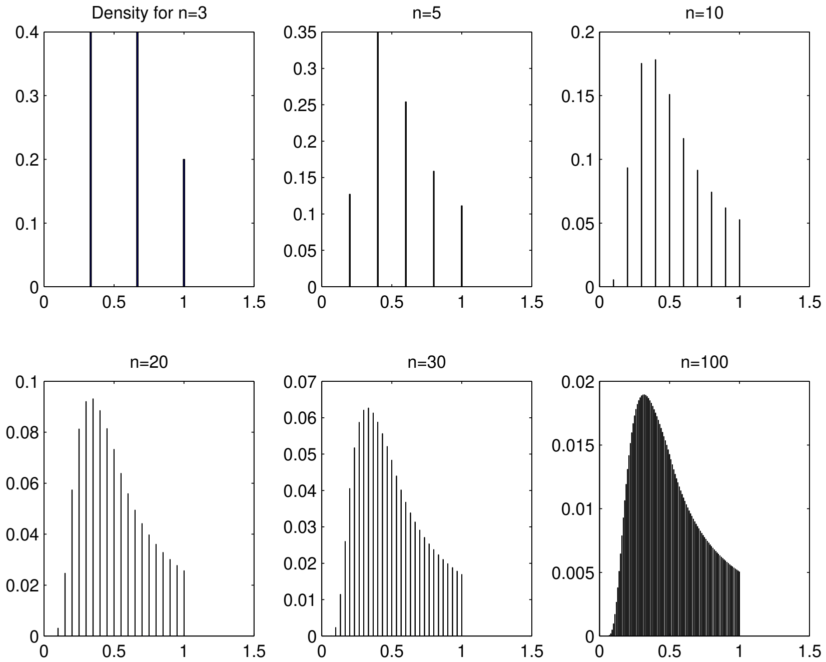

Figure 4: Density for longest excursion for different n.

4 Longest excursion of the Brownian bridge

In this section, we will give some asymptotic results for $G_{2n}(x)$. The first result concerns the recursive relation for the distribution of the longest excursion. This result leads to exact formulas for the asymptotic distribution, for certain intervals. This asymptotic distribution is the distribution for the longest excursion in the Brownian bridge, see Section 6. The main tool for achieving these results is Stirling's formula:

$$n! = \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}(1+\epsilon_n), \quad \epsilon_n \to 0, \ n \to \infty.$$

In the case of u_{2n}, for example, Stirling's formula gives

$$u_{2n} = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} = \frac{(2n)!}{(n!)^2}\left(\frac{1}{2}\right)^{2n} \sim \frac{1}{\sqrt{\pi n}}, \quad n \to \infty. \tag{7}$$

4.1 A recursive relation

Let

$$G_\infty(x) = \lim_{n\to\infty} G_{2n}(x), \quad 0 \le x \le 1,$$

and for $\frac{l}{n} \sim x$

$$g_\infty(x) = \frac{d}{dx}G_\infty(x) = \lim_{n\to\infty} n\mathbf{P}(L_{2n} = 2l | S_{2n} = 0), \quad 0 \le x \le 1.$$

Theorem 4.1.1 *For* $0 \le x \le 1$,

$$g_\infty(x) = \frac{1}{2x^{\frac{3}{2}}}\int_0^1 \frac{1}{\pi\sqrt{u(1-u)}} G_\infty\left(\frac{x}{(1-x)u}\right) G_\infty\left(\frac{x}{(1-x)(1-u)}\right) du.$$

Proof. The proof follows essentially from the asymptotic properties of equation (5). Start by multiplying each side by n. Remember that we let $n, l \to \infty$ so that $\frac{l}{n} \sim x$ for $0 \le x \le 1$. It is easily seen that the left hand side equals $g_\infty(x)$. The right hand side of (5) can now be written as

$$n\frac{f_{2l}^{(1)}}{u_{2n}} \sum_{j=0}^{n-l} n u_{2j} u_{2n-2j-2l} G_{2j}((l-1)/j) G_{2n-2l-2j}(l/(n-l-j))\frac{1}{n}.$$

With $v \sim \frac{j}{n}$, this is identified as a Riemann sum of the following integral

$$\frac{1}{2x^{\frac{3}{2}}}\int_0^{1-x} \frac{1}{\pi\sqrt{v(1-x-v)}} G_\infty\left(\frac{x}{v}\right) G_\infty\left(\frac{x}{1-x-v}\right) dv. \tag{8}$$

The convergence is seen in a few steps. The first part is that

$$n\frac{f_{2l}^{(1)}}{u_{2n}} = \frac{n}{2l-1}\frac{u_{2l}}{u_{2n}} \sim \frac{n}{2l}\frac{\frac{1}{\sqrt{\pi l}}}{\frac{1}{\sqrt{\pi n}}} = \frac{1}{2x^{\frac{3}{2}}},$$

by Stirling's formula and equation (3). Furthermore

$$u_{2j} \sim \frac{1}{\sqrt{\pi v n}}, \quad u_{2n-2l-2j} \sim \frac{1}{\sqrt{\pi n(1-x-v)}},$$

gives

$$n u_{2j} u_{2n-2l-2j} \sim \frac{1}{\pi\sqrt{v(1-x-v)}}.$$

The argument of $G_{2j}((l-1)/j)$ tends to $\frac{x}{v}$ through

$$\frac{(l-1)}{j} \sim \frac{l}{n}\frac{n}{j} = \frac{x}{v}.$$

The argument of $G_{2n-2l-2j}(l/(n-l-j))$ tends to $\frac{x}{1-x-v}$ through

$$\frac{l}{n-l-j} = \frac{\frac{l}{n}}{1-\frac{l}{n}-\frac{j}{n}} \sim \frac{x}{1-x-v}.$$

Finally $\frac{1}{n} = \Delta\left(\frac{j}{n}\right) \sim dv$. The proof finishes by the substitution $v = u(1-x)$ in equation (8).

$\square$

4.1.1 Numerical evaluation of the recursive relation

Theorem 4.1.1 gives a possibility to evaluate the distribution $G_{2n}(x)$ numerically. First, write the integral in Theorem 4.1.1 as

$$g_\infty(x) = \frac{1}{x^{\frac{3}{2}}}\int_0^{\frac{1}{2}} \frac{1}{\pi\sqrt{u(1-u)}} G_\infty\left(\frac{x}{(1-x)u}\right) G_\infty\left(\frac{x}{(1-x)(1-u)}\right) du. \quad (9)$$

A discretization with meshes Δx and $\frac{1}{M}$ of (9), writing $G(x)$ for $G_\infty(x)$, gives

$$\frac{G(x)-G(x-\Delta x)}{\Delta x} = \frac{1}{x^{3/2}} \sum_{j=0}^{M-1} \frac{G\left(\frac{x}{(1-x)\frac{j}{2M}}\right) G\left(\frac{x}{(1-x)(1-\frac{j}{2M})}\right)}{\pi\sqrt{\frac{j}{2M}(1-\frac{j}{2M})}} \frac{1}{2M}. \quad (10)$$

A quick study of (10) reveals that $G(x)$ just depends on $G(y)$ where $y > x$. This makes it possible to calculate $G(x)$ from the right, starting with $G(z) > 1$ for $z > 1$. A computer program using this structure is given in Appendix C. A run of the program gives the density and distribution in figure 5.

We can compare these results with those in table 1 and see that this numerical evaluation is a rough approximation when the mesh sizes are as

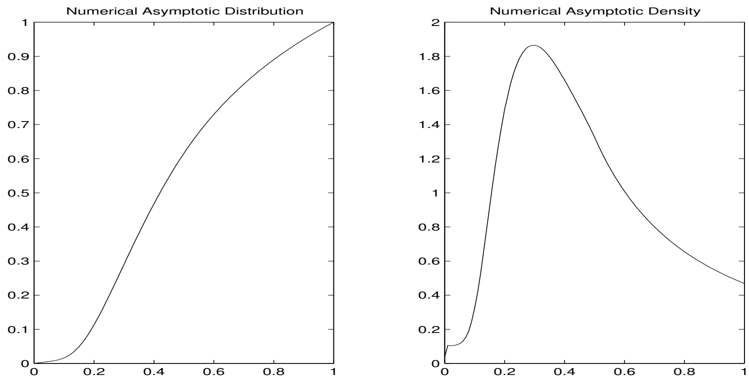

Figure 5: Distribution and density for longest excursion (mesh sizes are $\frac{1}{M} = \frac{1}{100}$ and $\Delta x = \frac{1}{100}$).

$x = m/n$	
$x = 3/4$	.8633
$x = 1/2$	.6288
$x = 1/4$	.2160

Table 2: Numerical evaluation of $G_{2n}(x)$ for different x

small as $\frac{1}{M} = \frac{1}{100}$ and $\Delta x = \frac{1}{100}$. Comparison with figure 4 casts doubt on the accuracy for small x for the density. Apparently, the discretization used here is not good enough to capture the steepness of the density curve for small x. Further comparisons are mentioned in Section 4.4.

4.2 Exact expressions for certain intervals

It turns out that it is possible to recover exact expressions for the asymptotic distribution of the longest excursion time by using the recursive relation and knowing the exact distribution for the interval $\frac{1}{2} \leq x \leq 1$. The exact distribution for the interval $\frac{1}{2} \leq x \leq 1$ can be attained in (at least) two ways. One way is to combine the result in Lemma 3.1.1 and (7) to get for $\frac{1}{2} \leq x \leq 1$

$$G_{2n}(x) = 2 - \frac{u_{2[nx]}}{u_{2n}} \sim 2 - \frac{\frac{1}{\sqrt{\pi n x}}}{\frac{1}{\sqrt{\pi n}}} = 2 - \frac{1}{\sqrt{x}}, \quad n \to \infty.$$

The other way to establish the result is just to examine the recursive relation when $\frac{1}{2} \leq x \leq 1$. Write Theorem 4.1.1 as

$$g_\infty(x) = \frac{1}{x^{\frac{3}{2}}} \int_0^{\frac{1}{2}} \frac{1}{\pi\sqrt{u(1-u)}} G_\infty\left(\frac{x}{(1-x)u}\right) G_\infty\left(\frac{x}{(1-x)(1-u)}\right) du.$$

This is possible, because the integrand is symmetric. Now, we realise that $\frac{x/(1-x)}{u} \geq 1$ and $\frac{x/(1-x)}{1-u} \geq 1$. This gives

$$g_\infty(x) = \frac{1}{x^{\frac{3}{2}}} \int_0^{\frac{1}{2}} \frac{1}{\pi\sqrt{u(1-u)}} du = \frac{1}{x^{\frac{3}{2}}} \frac{2}{\pi} \arcsin\sqrt{\frac{1}{2}} = \frac{1}{2x^{\frac{3}{2}}},$$

and

$$G_\infty(x) = 1 - \int_x^1 g_\infty(u)du = 2 - \frac{1}{\sqrt{x}}.$$

Theorem 4.2.1 *For $\frac{1}{2} \leq x \leq 1$, we have*

$$g_\infty(x) = \frac{1}{2x^{\frac{3}{2}}}, \quad G_\infty(x) = 2 - \frac{1}{\sqrt{x}}.$$

For $\frac{1}{3} \leq x \leq \frac{1}{2}$, we have

$$g_\infty(x) = \frac{1}{2x^{\frac{3}{2}}} \left[3 - \frac{4}{\pi} \arcsin\sqrt{\frac{x}{1-x}} - \frac{4}{\pi}\sqrt{\frac{1-2x}{x}}\right],$$

$$G_\infty(x) = 2 - \frac{3}{\sqrt{x}} + \frac{2}{\pi}\left[\frac{2}{\sqrt{x}} \arcsin\sqrt{\frac{x}{1-x}} + \frac{\sqrt{1-2x}}{x} + 2\arctan\sqrt{1-2x}\right].$$

Proof. The first part of the theorem is proved in the beginning of this section. In the proof of the second part, we integrate over the two intervals $0 \leq u \leq 1 - \frac{x}{1-x}$ and $1 - \frac{x}{1-x} \leq u \leq \frac{1}{2}$, where we have the following inequalities

$$1 \leq \frac{x}{(1-x)u}, \quad 0 \leq u \leq 1 - \frac{x}{1-x}, \quad \frac{1}{3} \leq x \leq \frac{1}{2}$$

$$1 \leq \frac{x}{(1-x)u}, \quad 1 - \frac{x}{1-x} \leq u \leq \frac{1}{2}, \quad \frac{1}{3} \leq x \leq \frac{1}{2}$$

$$\frac{1}{2} \leq \frac{x}{(1-x)(1-u)} \leq 1, \quad 0 \leq u \leq 1 - \frac{x}{1-x}, \quad \frac{1}{3} \leq x \leq \frac{1}{2}$$

$$1 \leq \frac{x}{(1-x)(1-u)}, \quad 1 - \frac{x}{1-x} \leq u \leq \frac{1}{2}, \quad \frac{1}{3} \leq x \leq \frac{1}{2}.$$

These inequalities gives

$$\begin{aligned} g_\infty(x) &= \frac{1}{x^{\frac{3}{2}}}\left[\int_0^{1-\frac{x}{1-x}} \frac{1}{\pi\sqrt{u(1-u)}} du + \int_{1-\frac{x}{1-x}}^{\frac{1}{2}} \frac{2 - \sqrt{\frac{(1-u)(1-x)}{x}}}{\pi\sqrt{u(1-u)}} du\right] \\ &= \frac{1}{2x^{\frac{3}{2}}}\left[3 - \frac{4}{\pi}\arcsin\sqrt{\frac{x}{1-x}} - \frac{4}{\pi}\sqrt{\frac{1-2x}{x}}\right]. \end{aligned}$$

The distribution is now easily obtained by

$$
\begin{aligned}
&G_\infty(x) = 1 - \int_x^1 g_\infty(u)du \\
&= 1 - \Bigg[\int_x^{1/2} \frac{1}{2u^{3/2}} \Bigg(3 - \frac{4}{\pi} \arcsin\sqrt{\frac{u}{1-u}} - \frac{4}{\pi}\sqrt{\frac{1-2u}{u}} \Bigg) du \\
&+ \int_{1/2}^1 \frac{1}{2u^{3/2}} du \Bigg] \\
&= 2 - \frac{3}{\sqrt{x}} + \frac{2}{\pi}\Bigg[\frac{2}{\sqrt{x}} \arcsin\sqrt{\frac{x}{1-x}} + \frac{\sqrt{1-2x}}{x} + 2\arctan\sqrt{1-2x} \Bigg].
\end{aligned}
$$

□

The results about the distribution in Theorem 4.2.1 are obtained in [7]: Gourdon (1998) by combinatorial methods. The results about the density in Theorem 4.2.1 are obtained in [9]: Pitman and Yor (1997). [9] extends the density results further to cover the interval $1/4 < x < 1$.

Theorem 4.2.2 *(Pitman and Yor) The density of the length V_1 of the longest excursion of a Brownian bridge is given by*

$$g_\infty(x) = q_1(x) - q_2(x) + q_3(x),$$

for $1/4 < x < 1$, where the $q_n(x)$ are

$$q_1(x) = \frac{1}{2} x^{-\frac{3}{2}},$$

$$q_2(x) = I(x \le \frac{1}{2}) \frac{1}{\pi} x^{-\frac{3}{2}} \Bigg(-\pi + 2\sqrt{\frac{1-2x}{x}} + 2\arcsin\sqrt{\frac{x}{1-x}} \Bigg),$$

$$q_3(x) = I(x \le \frac{1}{3}) \frac{3}{4\pi} x^{-\frac{3}{2}} \Bigg(2 + 2\pi + \frac{2}{x} - 8\sqrt{\frac{1-2x}{x}} - 8\arcsin\sqrt{\frac{x}{1-x}} \Bigg).$$

□

The density and distribution can now be plotted for $1/4 < x < 1$.

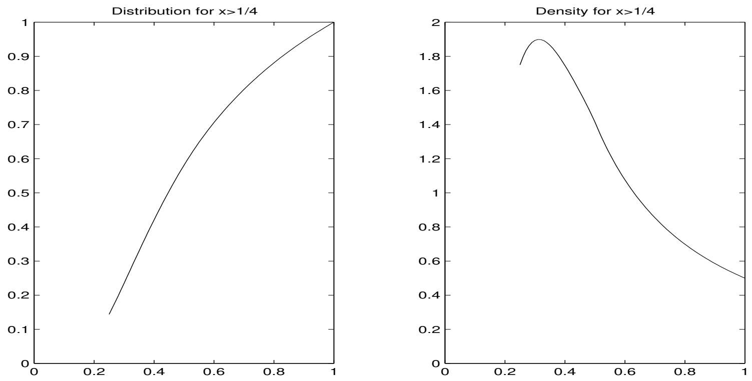

Figure 6: The distribution and density for the longest excursion in the interval $1/4 < x < 1$.

$x = m/n$	
$x = 3/4$	.8453
$x = 1/2$	.5858
$x = 1/4$	.1438*

Table 3: Exact evaluation of $G_{2n}(x)$ for different x. $*$ means that this value is attained by numerical integration using the density in Theorem 4.2.2.

4.3 An alternative representation of the distribution– a Kummer representation.

In this section, we will investigate the asymptotic distribution of the longest excursion using Laplace transforms. This will lead to an alternative representation of the distribution as an infinite sum involving the zeros of a certain Kummer function. The following formula, due to Rosén, is given in [11] with a hint to a proof. We will sketch a proof of the representation:

$$G_\infty(x) = \frac{2\pi}{\sqrt{x}} \sum_{k=-\infty}^{\infty} (-s_k) \exp(s_k(1 + x^{-1})), \quad 0 < x < 1. \tag{11}$$

The numbers s_k range over the zeros of the confluent hypergeometric function or Kummer function $\Phi(1, 1/2; s)$, see [2]: Abramovitz and Stegun (1972) and Appendix A.

First, we study the representation. Then, we sketch a proof based on Laplace transforms.

4.3.1 Study of the representation.

First, we give the representation in a real form. Then, we plot $G_\infty(x)$ by truncation of the infinite sum. $g_\infty(x)$ is also studied. Then, we compare the truncated representation of $g_\infty(x)$ with the true function in the interval $1/4 < x < 1$.

Using $s_k = -a_k + b_k i$, see Appendix A, (11) can be written as

$$\begin{aligned}
G_\infty(x) &= \frac{2\pi}{\sqrt{x}}\Big(-s_0 \exp(s_0(1+1/x)) \\
&+ 2\sum_{k=1}^{\infty} a_k \cos(b_k(1+1/x)) \exp(-a_k(1+1/x)) \\
&+ 2\sum_{k=1}^{\infty} b_k \sin(b_k(1+1/x)) \exp(-a_k(1+1/x))\Big), \quad 0 < x < 1.
\end{aligned}$$

We can also differentiate (11) to get, for $0 < x < 1$

$$\begin{aligned}
g_\infty(x) &= \frac{-\pi}{x^{3/2}} \sum_{k=-\infty}^{\infty} (-s_k) \exp(s_k(1+1/x)) \\
&+ \frac{2\pi}{x^{5/2}} \sum_{k=-\infty}^{\infty} (-s_k)^2 \exp(s_k(1+1/x)) \\
&= \frac{-\pi}{x^{3/2}}\Big(-s_0 \exp(s_0(1+1/x)) \\
&+ 2\sum_{k=1}^{\infty} a_k \cos(b_k(1+1/x)) \exp(-a_k(1+1/x)) \\
&+ 2\sum_{k=1}^{\infty} b_k \sin(b_k(1+1/x)) \exp(-a_k(1+1/x))\Big) \\
&+ \frac{2\pi}{x^{5/2}}\Big((-s_0)^2 \exp(s_0(1+1/x)) \\
&+ 2\sum_{k=1}^{\infty} (a_k^2 - b_k^2) \exp(-a_k(1+1/x)) \cos(b_k(1+1/x)) \\
&+ 4\sum_{k=1}^{\infty} a_k b_k \exp(-a_k(1+1/x)) \sin(b_k(1+1/x))\Big). \qquad (12)
\end{aligned}$$

Now, we are able to plot $G_\infty(x)$ and $g_\infty(x)$ truncated at different k, i.e. involving $2k+1$ zeros in the representation, see figure 7. We can zoom in to

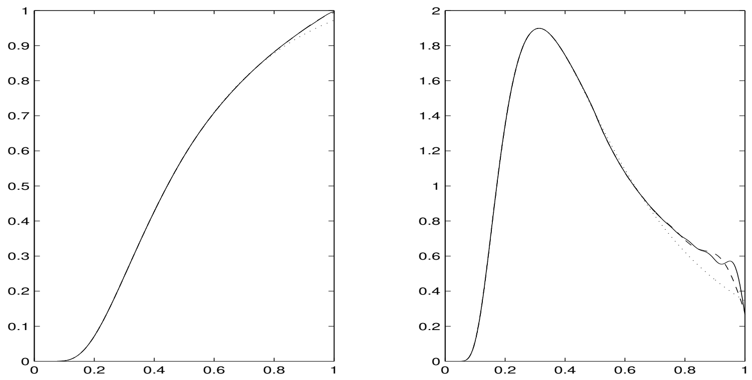

Figure 7: $G_\infty(x)$ and $g_\infty(x)$ truncated at k=0 (..), k=4 (–) and k=10 (-.).

get a closer look at the behaviour, see figure 8. Next, we will compare the

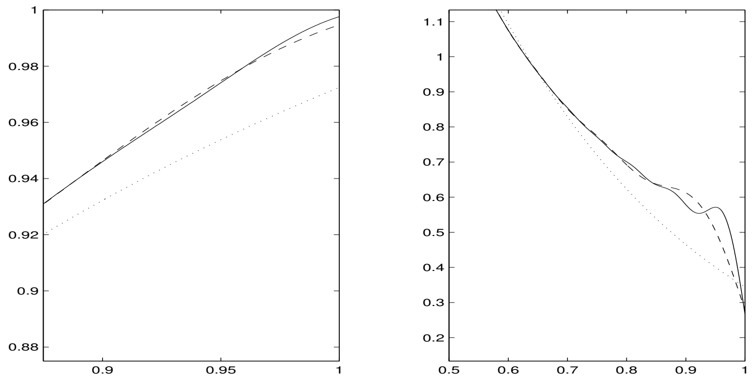

Figure 8: Zoom on $G_\infty(x)$ and $g_\infty(x)$ truncated at k=0 (..), k=4 (- -) and k=10 (-).

truncated $g_\infty(x)$ with the true $g_\infty(x)$, see figures 9 and 10.

Truncating at k=0 gives a remarkably accurate approximation for $x \leq 1/2$. A conclusion is that we have the exact distribution down to $x = 1/4$ and almost the exact distribution in the interval $0 < x < 1/4$ by the infinite series representation, truncated at the first zero.

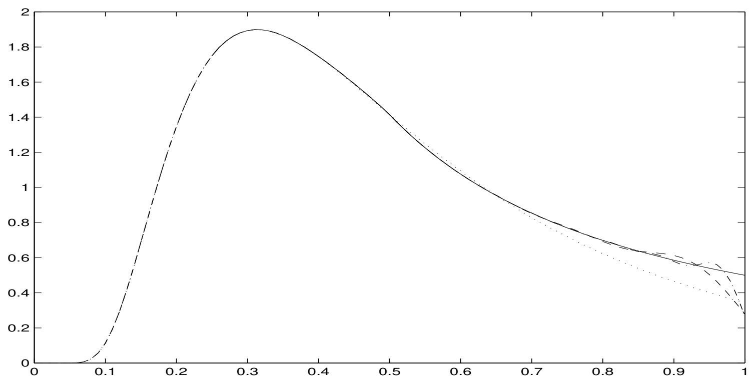

Figure 9: Comparison of true $g_\infty(x)$ and truncated $g_\infty(x)$ at k=0 (..), k=4 (- -) and k=10 (-.).

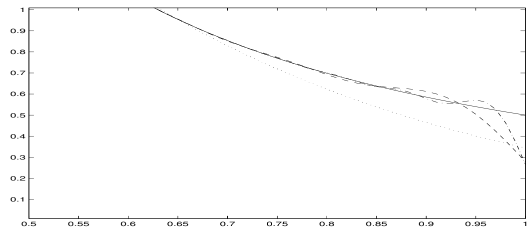

Figure 10: Zoomed comparison of true $g_\infty(x)$ and truncated $g_\infty(x)$ at k=0 (..), k=4 (- -) and k=10 (-.).

The fast convergence in (11) can be seen by investigating the terms in the expression (12). First, consider the terms

$$term1, k = 2\frac{-\pi}{x^{3/2}} a_k \cos(b_k(1+1/x)) \exp(-a_k(1+1/x)).$$

$$term2, k = 2\frac{-\pi}{x^{3/2}} b_k \sin(b_k(1+1/x)) \exp(-a_k(1+1/x)).$$

$$term3, k = 2\frac{2\pi}{x^{5/2}} ({a_k}^2 - {b_k}^2) \cos(b_k(1+1/x)) \exp(-a_k(1+1/x)).$$

$$term4, k = 4\frac{2\pi}{x^{5/2}} (a_k b_k) \sin(b_k(1+1/x)) \exp(-a_k(1+1/x)).$$

It follows that the terms approach zero rapidly when x approaches zero, because a_k is positive. Another remark is that the terms approach zero with increasing k, because a_k is increasing. Note that the terms may be both positive and negative depending on the argument for the cosine and sine functions, which leads to a cancellation in the sum. Let us compare the values of the terms for different k with the term containing s_0, which is

$$term0 = \frac{\pi}{x^{3/2}}(s_0) \exp(s_0(1+1/x)) + \frac{2\pi}{x^{5/2}}(-s_0)^2 \exp(s_0(1+1/x))$$

Tables 4 and 5 shows the comparisons for $x = 0.25$ and $x = 0.5$.

k	$term0$	$term1, k$	$term2, k$	$term3, k$	$term4, k$
0	1.75001977578388	0	0	0	0
1	0	-1.1130e-07	-9.1278e-08	-4.7551e-06	6.2053e-06
2	0	3.4412e-11	-3.5816e-09	7.3897e-09	2.9708e-07
3	0	4.8904e-11	-2.6628e-10	2.3015e-08	2.4508e-08
4	0	9.7940e-12	-3.4423e-11	7.8622e-09	3.3944e-09
5	0	2.3829e-12	-6.2237e-12	2.8764e-09	6.4579e-10
6	0	7.0547e-13	-1.3782e-12	1.1859e-09	1.4885e-10
7	0	2.4446e-13	-3.3813e-13	5.4326e-10	3.7737e-11
8	0	9.5930e-14	-8.1581e-14	2.7143e-10	9.3604e-12
9	0	4.1589e-14	-1.3933e-14	1.4561e-10	1.6372e-12
10	0	1.9554e-14	3.3696e-15	8.2841e-11	-4.0433e-13
Total sum	1.75002138886890	-	-	-	-
True value	1.75002138902061	-	-	-	-

Table 4: Comparison of terms in Kummer representation for $x = 0.25$.

k	$term0$	$term1, k$	$term2, k$	$term3, k$	$term4, k$
0	1.41444069778343	0	0	0	0
1	0	-2.1006e-04	-9.7553e-05	-4.4871e-03	3.3160e-03
2	0	-9.3460e-06	-3.2956e-05	-1.0035e-03	1.3668e-03
3	0	-9.5875e-07	-1.0484e-05	-2.2560e-04	4.8244e-04
4	0	-1.0917e-07	-4.2269e-06	-4.3817e-05	2.0841e-04
5	0	8.8644e-09	-2.0168e-06	5.3501e-06	1.0464e-04
6	0	2.2091e-08	-1.0838e-06	1.8567e-05	5.8529e-05
7	0	1.8743e-08	-6.3557e-07	2.0827e-05	3.5466e-05
8	0	1.3893e-08	-3.9829e-07	1.9655e-05	2.2849e-05
9	0	1.0020e-08	-2.6294e-07	1.7541e-05	1.5449e-05
10	0	7.2402e-09	-1.8102e-07	1.5337e-05	1.0860e-05
Total sum	1.41402910029131	-	-	-	-
True value	1.41421356237309	-	-	-	-

Table 5: Comparison of terms in Kummer representation for $x = 0.5$.

We can see in the tables that the expression with just the first zero is a very sharp approximation of the true values for $x = 0.25$ and $x = 0.5$. A closer look at the terms involved in the sums also reveals that we get an even better approximation as x approaches zero.

The tables 4 and 5 lists the first 10 terms in the sums of the expression (12). It is possible to get upper bounds for the remaining sum. Wendel ([11]) states that, for $k \to \infty$

$$a_k \sim 3\ln(k), \quad b_k \sim 2\pi k.$$

It is possible to insert this in the sums and estimate the magnitude of the remaining terms, but we do not do it here.

Another way to illustrate the influence on the density of the first zero relative to the next 20 zeros is to plot the density using the first zero versus to plot the contribution to the density using zeros 2-21. Note that the superposition of these two curves constitutes the curve with the first 21 zeros (k=10) in figure 7.

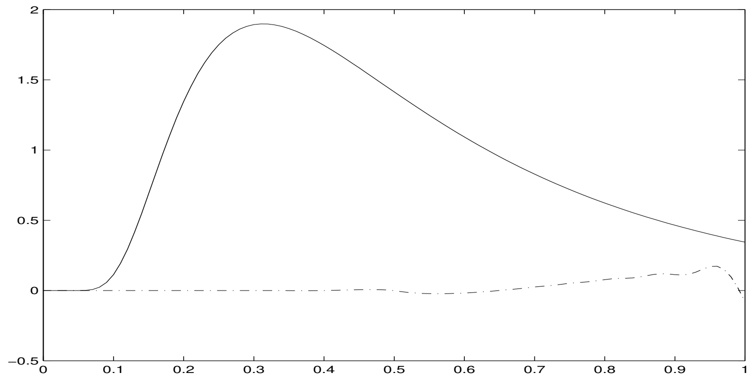

Figure 11: (-) represents density with Kummer representation with the first zero. (-.) represents the influence of the next 20 zeros.

We can see that zeros 2-21 seems to have no impact for $x < 0.5$. This leads to the conclusion that the representation with the first zero is a sufficient approximation for $x < 0.5$.

4.3.2 Proof of the representation.

A hint to a proof of the representation (11) is given in [11]. Based on Holst:[8], we will sketch a different proof using Laplace transforms and calculus of residues. Let $Y_1, Y_2, \ldots$ be successive excursion times in a simple symmetric random walk, see Section 2.4, and set $Z_k = Y_k/2$. Let

$$N_{2n} = \max\{l : Z_1 + \cdots + Z_l \leq n\},$$

and consider the sum

$$\sum_{j=1}^{N_{2n}} h\left(\frac{Y_j}{2n}\right), \tag{13}$$

where h is a suitable function such that

$$0 = h(0) \leq h(y) < \infty,$$

and for $s \geq 0$

$$\int_0^\infty \frac{1 - e^{-sh(y)+ity}}{y^{3/2}} dy$$

is finite. We will in the following only consider $h(y) = I(y > x)$, where $0 < x < 1$. Let also $V_1 \geq V_2 \geq \ldots$ be the ordered excursion lengths of a Brownian bridge, such that $\sum_{j=1}^{\infty} V_j = 1$. Now, consider the Laplace transform of the sum (13).

Lemma 4.3.1 *For $s > 0$, $h(y) = I(y > x)$ and $0 < x < 1$*

$$\begin{aligned}
\mathbf{E}\left(e^{-s\sum_{j=1}^{N_{2n}} h(Y_j/2n)}\middle| S_{2n} = 0\right) &\to \int_{-\infty}^{\infty} \frac{e^{-it}}{\int_0^\infty \frac{(1-e^{-sh(y)+ity})}{y^{3/2}}dy}dt \\
&= \mathbf{E}\left(e^{-s\sum_{j=1}^{\infty} h(V_j)}\right),
\end{aligned}$$

as $n \to \infty$. We also have (for $s = 0$)

$$\int_{-\infty}^{\infty} \frac{e^{-it}}{\int_0^\infty \frac{(1-e^{ity})}{y^{3/2}}dy}dt = 1.$$

Sketch of a proof. Let $s > 0$.

$$\begin{aligned}
&\mathbf{E}\left(e^{-s\sum_{j=1}^{N_n} h(Z_j/n)}\middle| S_{2n} = 0\right) = \\
&= \frac{1}{\mathbf{P}(S_{2n} = 0)} \sum_{r=1}^{n} \mathbf{E}\left(e^{-s\sum_{j=1}^{r} h(Z_j/n)} I_{\{\sum_{j=1}^{r} Z_j = n\}}\right) \\
&= \left\{\text{By orthogonality of } e^{int}\right\} \\
&= \frac{1}{2\pi\mathbf{P}(S_{2n} = 0)} \sum_{r=1}^{n} \int_{-\pi}^{\pi} e^{-itn}\mathbf{E}\left(e^{-s\sum_{j=1}^{r} h(\frac{Z_j}{n})+it\sum_{j=1}^{r} Z_j}\right)dt \\
&= \left\{\text{By substitution } \tfrac{u}{n} = t \text{ and independent } Z_j\right\} \\
&= \frac{1}{2\pi n\mathbf{P}(S_{2n} = 0)} \sum_{r=1}^{n} \int_{-\pi n}^{\pi n} e^{-iu}\left(\mathbf{E}\left(e^{-sh(\frac{Z}{n})+\frac{iuZ}{n}}\right)\right)^r du \\
&= \left\{\text{Change summation and integration order, } u = t\right\} \\
&= \frac{1}{2\pi n\mathbf{P}(S_{2n} = 0)} \int_{-\pi n}^{\pi n} e^{-it} \sum_{r=1}^{n} \left(\mathbf{E}\left(e^{-sh(\frac{Z}{n})+\frac{itZ}{n}}\right)\right)^r dt \\
&= \left\{\text{By known series expansion}\right\} \\
&= \frac{(2\pi n)^{-1}}{\mathbf{P}(S_{2n} = 0)} \int_{-\pi n}^{\pi n} e^{-it} \frac{\mathbf{E}\left(e^{-sh(\frac{Z}{n})+\frac{itZ}{n}}\right) - \left(\mathbf{E}\left(e^{-sh(\frac{Z}{n})+\frac{itZ}{n}}\right)\right)^{n+1}}{1-\mathbf{E}\left(e^{-sh(\frac{Z}{n})+\frac{itZ}{n}}\right)}dt \\
&\to \left\{\text{See below}\right\} \to \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-it}}{\int_0^\infty \frac{(1-e^{-sh(y)+ity})}{2\sqrt{\pi}y^{3/2}}dy}dt, \quad n \to \infty.
\end{aligned}$$

We will now see why the last equality tends to the last expression. First, look at the numerator in the integral. Let $\mathbf{P}(Z=j)=f_{2j}$. As $n\to\infty$:

$$\mathbf{E}\left(e^{-sh(\frac{Z}{n})+\frac{itZ}{n}}\right)=\sum_{j=1}^{\infty}f_{2j}e^{-sh(\frac{j}{n})+\frac{itj}{n}}\to\sum_{j=1}^{\infty}f_{2j}=1,\quad n\to\infty,$$

and for fixed $s>0$

$$\left(\mathbf{E}\left(e^{-sh(\frac{Z}{n})+\frac{itZ}{n}}\right)\right)^{n+1}=\left(\sum_{j=1}^{\infty}f_{2j}e^{-sh(\frac{j}{n})+\frac{itj}{n}}\right)^{n+1}\to 0.$$

These two results gives that the numerator tends to 1. Let us now look at the denominator. Let $n,j\to\infty$, such that $\frac{j}{n}\sim y$. Then

$$\begin{aligned}
\sqrt{n}\left(1-\mathbf{E}\left(e^{-sh(\frac{Z}{n})+\frac{itZ}{n}}\right)\right) &= \sqrt{n}\mathbf{E}\left(1-\left(e^{-sh(\frac{Z}{n})+\frac{itZ}{n}}\right)\right)\\
&= \sqrt{n}\sum_{j=1}^{\infty}\left(1-e^{-sh(\frac{j}{n})+\frac{itj}{n}}\right)f_{2j}\\
&\sim \left\{\text{Equations (3) and (7)}\right\}\\
&\sim \sqrt{n}\sum_{j=1}^{\infty}\left(1-e^{-sh(\frac{j}{n})+\frac{itj}{n}}\right)\frac{1}{2\pi j^{3/2}}\\
&= \sum_{j=1}^{\infty}\left(1-e^{-sh(\frac{j}{n})+\frac{itj}{n}}\right)\frac{1}{2\pi(\frac{j}{n})^{3/2}}\frac{1}{n}\\
&\sim \left\{\text{A riemann sum, } y\sim\tfrac{j}{n}.\right\}\\
&\sim \int_0^{\infty}(1-e^{-sh(y)+ity})\frac{1}{2\pi y^{3/2}}dy.
\end{aligned}$$

Finally, we have

$$\mathbf{P}(S_{2n}=0)=u_{2n}\sim\frac{1}{\sqrt{\pi n}},\quad n\to\infty,$$

which gives the expression before the integral sign.

To show continuity of the Laplace transform, we have to show that

$$\int_{-\infty}^{\infty}\frac{e^{-it}}{\int_0^{\infty}\frac{(1-e^{ity})}{y^{3/2}}dy}dt=1.$$

First, look at

$$\int_0^{\infty}\frac{(1-e^{ity})}{y^{3/2}}dy=\left[-\frac{2(1-e^{ity})}{\sqrt{y}}\right]_0^{\infty}+2it\int_0^{\infty}\frac{e^{ity}}{\sqrt{y}}dy$$

$$
\begin{aligned}
&= 2it\Big(\int_0^\infty \frac{\cos ty}{\sqrt{y}}dy + i\int_0^\infty \frac{\sin ty}{\sqrt{y}}dy\Big)\\
&= 2it\Big(\int_0^\infty \frac{\cos |t|y}{\sqrt{y}}dy + i\ \mathrm{sgn}(t)\int_0^\infty \frac{\sin |t|y}{\sqrt{y}}dy\Big)\\
&= 2i\frac{t}{|t|}\Big(\sqrt{(|t|)}\int_0^\infty \frac{\cos x}{\sqrt{x}}dx + i\ \mathrm{sgn}(t)\sqrt{(t)}\int_0^\infty \frac{\sin x}{\sqrt{x}}dx\Big)\\
&= \Big\{\text{Known integrals, see for instance [10]}\Big\}\\
&= 2i\ \mathrm{sgn}(t)\sqrt{|t|}(\sqrt{\frac{\pi}{2}} + i\ \mathrm{sgn}(t)\sqrt{\frac{\pi}{2}})\\
&= 2\sqrt{\pi}\sqrt{-it}. \qquad (14)
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\frac{1}{2\sqrt{\pi}}\int_{-\infty}^\infty \frac{e^{-it}}{\sqrt{-it}}dt\\
&= \frac{1}{2\sqrt{\pi}}\int_{-\infty}^\infty \frac{e^{p1}}{\sqrt{p}}dp\\
&= \Big\{\text{Inverse Laplace tansform of } \tfrac{1}{\sqrt{p}} \text{ is } \tfrac{1}{\sqrt{\pi}}\tfrac{1}{\sqrt{t}}\Big\}\\
&= 1.
\end{aligned}
$$

This continuity of the Laplace transform shows that the limit distribution is a proper distribution. For a statement of the continuity theorem for Laplace transforms, see [6]: Feller II (1971:p431).

The convergence

$$
\mathbf{E}\Big(e^{-s\sum_{j=1}^{N_{2n}} h(Y_j/2n)}\Big| S_{2n} = 0\Big) \to \mathbf{E}\Big(e^{-s\sum_{j=1}^{\infty} h(V_j)}\Big),
$$

is discussed in Section 6.

$\square$

Theorem 4.3.1 *For $0 < x < 1$, set $M = \sum_{j=1}^\infty I(V_j > x)$. Then,*

$$
\mathbf{P}(M = j) = \frac{1}{2\sqrt{x}i}\int_{-i\infty}^{i\infty} e^{s(\frac{1}{x}+1)}\frac{(\Phi(1,\frac{1}{2},s) - e^s\sqrt{\pi s})^j}{(\Phi(1,\frac{1}{2},s))^{j+1}}ds.
$$

Proof. Set $z = e^{-s}$ and recall that $h(y) = I(y > x)$. Using Lemma 4.3.1, we get

$$
\mathbf{E}(z^M) = \int_{-\infty}^\infty \frac{e^{-it}}{\int_0^\infty \frac{(1-e^{-sh(y)+ity})}{y^{3/2}}dy}dt
$$

$$
\begin{aligned}
&= \left\{\text{Set } t = u/x \text{ and } y = ax\right\} \\
&= \frac{x^{3/2}}{x^2}\int_{-\infty}^{\infty}\frac{e^{-iu/x}}{\int_0^\infty \frac{(1-e^{-sh(ax)+i\frac{u}{x}ax})}{a^{3/2}}da}du \\
&= \left\{\text{Set } u = t \text{ and } a = y\right\} \\
&= \frac{1}{\sqrt{x}}\int_{-\infty}^{\infty}\frac{e^{-it/x}}{\int_0^\infty \frac{(1-e^{-sh(yx)+ity})}{y^{3/2}}dy}dt \\
&= \frac{1}{\sqrt{x}}\int_{-\infty}^{\infty}\frac{e^{-it/x}}{\int_0^\infty \frac{(1-z^{I(yx>x)}e^{ity})}{y^{3/2}}dy}dt \\
&= \frac{1}{\sqrt{x}}\int_{-\infty}^{\infty}\frac{e^{-it/x}}{\int_0^\infty \frac{dy}{y^{3/2}} - \int_0^1 \frac{e^{ity}}{y^{3/2}}dy - \int_1^\infty \frac{ze^{ity}}{y^{3/2}}dy}dt \\
&= \left\{\int_1^\infty \frac{dy}{y^{3/2}} = 2\right\} \\
&= \frac{1}{\sqrt{x}}\int_{-\infty}^{\infty}\frac{e^{-it/x}}{(2+\int_0^1 \frac{1-e^{ity}}{y^{3/2}}dy) - z(2+\int_1^\infty \frac{e^{ity}-1}{y^{3/2}}dy)}dt \\
&= \left\{\text{Expand the expression in a Taylor series around } z = 0\right\} \\
&= \frac{1}{2\sqrt{x}}\sum_{j=0}^{\infty} z^j \int_{-\infty}^{\infty} e^{\frac{-it}{x}}\frac{(1-\frac{1}{2}\int_1^\infty \frac{1-e^{ity}}{y^{3/2}}dy)^j}{(1+\frac{1}{2}\int_0^1 \frac{1-e^{ity}}{y^{3/2}}dy)^{j+1}}dt
\end{aligned}
$$

The last expression and the definition of a probability generating function gives

$$
\mathbf{P}(M = j) = \frac{1}{2\sqrt{x}}\int_{-\infty}^{\infty} e^{\frac{-it}{x}}\frac{(1-\frac{1}{2}\int_1^\infty \frac{1-e^{ity}}{y^{3/2}}dy)^j}{(1+\frac{1}{2}\int_0^1 \frac{1-e^{ity}}{y^{3/2}}dy)^{j+1}}dt. \qquad (15)
$$

But,

$$
\begin{aligned}
& 1 + \frac{1}{2}\int_0^1 \frac{1-e^{-sy}}{y^{3/2}}dy \\
&= \left\{\text{Integration by parts}\right\} \\
&= 1 + \left[-\frac{1-e^{-sy}}{\sqrt{y}}\right]_0^1 + \int_0^1 \frac{se^{-sy}}{\sqrt{y}}dy \\
&= \left\{1 - e^{-sy} = \sqrt{y}(s\sqrt{y} + O(y^{3/2}))\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= e^{-s} + s\int_0^1 \frac{e^{-sy}}{\sqrt{y}}dy \\
&= \left\{\text{Integration by parts}\right\} \\
&= e^{-s} + \left[2\sqrt{y}se^{-sy}\right]_0^1 + 2s^2\int_0^1 \sqrt{y}e^{-sy}dy \\
&= \left\{\text{Integration by parts}\right\} \\
&= e^{-s} + 2se^{-s} + \left[2s^2\frac{2}{3}y^{3/2}e^{-sy}\right]_0^1 + \frac{4}{3}s^3\int_0^1 y^{3/2}e^{-sy}dy \\
&= \left\{\text{Continue integration by parts}\right\} \\
&= e^{-s}\left(1 + \frac{s}{\frac{1}{2}} + \frac{s^2}{\frac{1}{2}\frac{3}{2}} + \frac{s^3}{\frac{1}{2}\frac{3}{2}\frac{5}{2}} + \cdots\right) \\
&= e^{-s}\Phi(1, \frac{1}{2}, s). \qquad (16)
\end{aligned}
$$

More about $\Phi(1, \frac{1}{2}, s)$ is in Appendix A. The last result, (14), (15) and the substitution $-it = s$ gives the desired expression.

□

Remark 4.3.1 Because of the property that $V_1 \geq V_2 \geq \ldots$, we have

$$
\mathbf{P}(M = 0) = \mathbf{P}(\bigcap_{n=1}^{\infty}\{V_n < x\}) = \mathbf{P}(V_1 < x) = \frac{1}{2\sqrt{x}i}\int_{-i\infty}^{i\infty} \frac{e^{s(\frac{1}{x}+1)}}{\Phi(1, \frac{1}{2}, s)}ds,
$$

and for $j > 0$

$$
\mathbf{P}(M = j) = \mathbf{P}(\{\bigcap_{n=1}^{j}\{V_n > x\}\}\bigcap\{\bigcap_{n=j+1}^{\infty}\{V_n < x\}\}) = \mathbf{P}(V_j > x > V_{j+1}),
$$

and

$$
\mathbf{P}(M \leq j - 1) = \mathbf{P}(V_j \leq x).
$$

□

Remark 4.3.2 Theorem 4.3.1 gives a way of calculating the distribution of the n:th longest excursion. We need to evaluate integrals like

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\frac{(\Phi(1,\frac{1}{2},s)-e^{s}\sqrt{\pi s})^{j}}{(\Phi(1,\frac{1}{2},s))^{j+1}}ds.$$

This can be done by calculus of residues. We can also try to solve the integrals numerically by numerical inversion of Laplace transforms, see Appendix B. □

Calculus of residues gives

Corollary 4.3.1

$$\mathbf{P}(V_1 \le x) = G_\infty(x) = \frac{2\pi}{\sqrt{x}}\sum_{k=-\infty}^{\infty}(-s_k)\exp(s_k(1+x^{-1})),\ 0<x<1.$$

Proof. By using Remark 4.3.1, we get

$$\begin{aligned}\mathbf{P}(V_1 \le x) &= \frac{1}{2\sqrt{x}i}\int_{-i\infty}^{i\infty}\frac{e^{s(\frac{1}{x}+1)}}{\Phi(1,\frac{1}{2},s)}ds\\ &= \lim_{A\to\infty}\frac{1}{2\sqrt{x}}\frac{1}{i}\int_{-iA}^{iA}\frac{e^{\frac{s}{x}}}{(1+\frac{1}{2}\int_0^1\frac{1-e^{-sy}}{y^{3/2}}dy)}ds\\ &= \lim_{A\to\infty}\frac{1}{2\sqrt{x}}\frac{1}{i}\int_{-iA}^{iA}\frac{f(s)}{g(s)}ds. \qquad (17)\end{aligned}$$

But,

$$g(s) = 1+\frac{1}{2}\int_0^1\frac{1-e^{-sy}}{y^{3/2}}dy = e^{-s}\Phi(1,\frac{1}{2},s),$$

and

$$\frac{dg(s)}{ds} = \frac{1}{2}\int_0^1\frac{e^{-sy}}{\sqrt{y}}dy = e^{-s}(\Phi(1,\frac{1}{2},s)-1)/(2s).$$

We can now use the residue theorem to solve the integral in (17). Let $\{s_k\}_{k=-\infty}^{\infty}$ be the zeros to $\Phi(1,\frac{1}{2},s)$. Consider a contour Γ consisting of the parts $-iA < s < iA$ and $s = Ae^{i\Theta}$ $(\frac{\pi}{2}\le\Theta\le\frac{3\pi}{2})$. The only poles of the function $g(s)$ in the area contained in Γ are those given by $\Phi(1,\frac{1}{2},s)$. Note that all zeros to $\Phi(1,\frac{1}{2},s)$ have negative real parts.

The semicircular part of Γ tends to zero as A tends to infinity. We need to chose a sequence of A in a way so that the poles of the Kummer function are avoided (see [4]: Doetsch (1958)). This is possible, because the zeros do not cluster (see appendix A). For a full investigation of this, one needs a detailed examination of the behaviour of the Kummer function in the neighbourhood of $\pm i\infty$, but we do not do it here.

Now, it is possible to apply the residue theorem to solve the integral. First, we calculate the residues of the poles;

$$\mathrm{Res}\{\frac{f(s)}{g(s)}, s_k\} = \frac{f(s_k)}{g'(s_k)} = \frac{e^{s_k/x}}{e^{-s_k}(0-1)/(2s_k)}.$$

For $0 < x < 1$,

$$\begin{aligned} \lim_{A\to\infty} \frac{2\pi}{2\sqrt{x}} \frac{1}{2\pi i} \int_{-iA}^{iA} \frac{f(s)}{g(s)} ds &= \frac{2\pi i}{2\sqrt{x} i} \sum_{k=-\infty}^{\infty} \frac{e^{s_k/x}}{e^{-s_k}(0-1)/(2s_k)} \\ &= \frac{2\pi}{\sqrt{x}} \sum_{k=-\infty}^{\infty} (-s_k) \exp(s_k(1+x^{-1})). \end{aligned}$$

□

4.4 Comparisons of the different representations of the distribution of the longest excursion

This section explores the similarities and dissimilarities between the different representations of the density of the longest excursion in a random walk and in a Brownian bridge. There are four different representations. The first one is the explicit expression for the density of the Brownian bridge, for $x > 1/4$ (see Section 4.2). The second representation is the true density for the random walk conditioned on the event $S_{2n} = 0$ (Section 3.2). In this case we choose $n = 100$, i.e. 200 steps. The third representation is the Kummer series representation in Section 4.3. We choose to truncate this representation for 21 zeros, i.e. $k = 10$ in our previous notation. The last representation is the numerically apprehended density from the recursive relation in Section 4.1.1. We choose the mesh sizes equal to $1/100$.

The first interesting comparison is to compare the exact density of the Brownian bridge, which is the asymptotic density of the exact density for the scaled tied down random walk, with the density of the random walk. In order to compare these two densities, we boost up the density of the random walk by a factor $n = 100$ to get the same scaling. We can see that the densities are very close in the interval $x > 1/4$, figure 12. Now, it is interesting to see how the density of the random walk behaves for $x < 1/4$. We know from Section 4.3 that the Kummer series representation is almost exact for these values of x, because of the extremely fast convergence (even for low orders of the truncation). We can see that the random walk follows the Kummer representation very well, figure 13. This means that the exact density of the random walk for $n = 100$ is a surprisingly good approximation of the asymptotic density in the whole interval $0 < x < 1$. Now, we look at how the numerics of the recursive relation representation works.

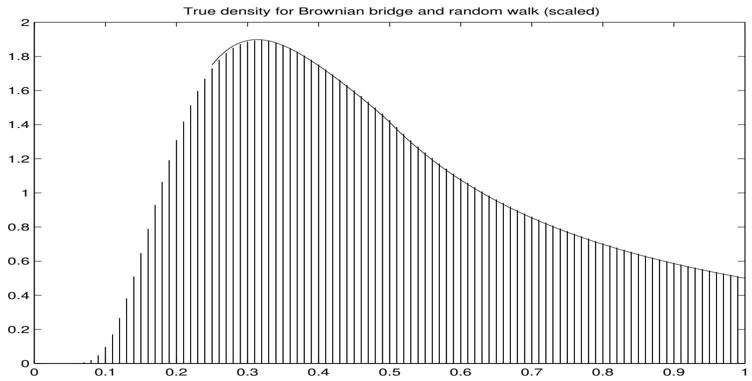

Figure 12: Comparison between the true density ($x > 1/4$) for the Brownian bridge and the true density for a tied down random walk with 200 steps for the longest excursion.

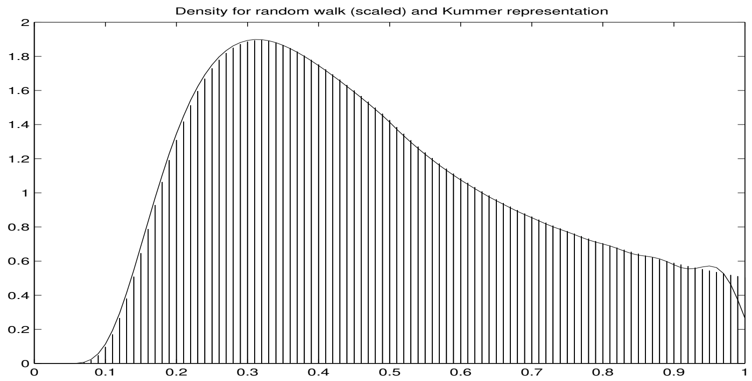

Figure 13: Comparison between the Kummer density representation for the Brownian bridge (truncated at 21 zeros) and the true density for a tied down random walk with 200 steps for the longest excursion.

We can see, that the numerics seems really poor for the chosen mesh sizes, figure 14. The convergence to the true density for the numerics is slow with the numerical method used here. Maybe, the convergence can be improved with a different numerical method. It also seems like the convergence for small x is even slower than the convergence in general, by inspection of the figure.

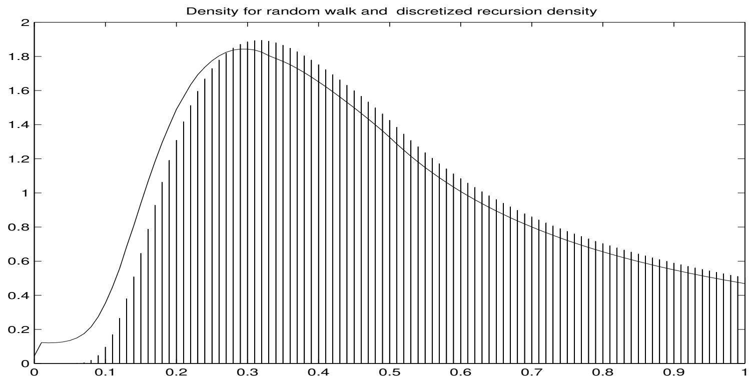

Figure 14: Comparison between the recursive relation density representation for the Brownian bridge (mesh sizes=1/100) and the true density for a tied down random walk with 200 steps for the longest excursion.

4.5 Moments and mode of the longest excursion

The mode is easily obtained from the exact expressions of the density in Section 4.2. We can also get a good estimate of the moments by using the exact expression for the density in the interval $1/2 < x < 1$ and the Kummer representation in the interval $0 < x < 1/2$. The mean and variance of the longest excursion are given in [7] by the asymptotic formulas

$$\mathbf{E}(L_n) \sim c_1 n, \text{ where } c_1 = 0.4834983\ldots,$$

and

$$\mathbf{Var}(L_n) \sim c_2 n^2, \text{ where } c_2 = 0.0494698\ldots.$$

Looking at figure 6, we can see that the mode is in the interval $1/4 < x < 1/3$. By differentiating the density and putting the derivative equal to zero in the interval we get:

$$\mathbf{Mode} = 0.3135097128\ldots$$

The moments can be estimated from the exact expression for the density in the interval $1/2 < x < 1$ and the Kummer representation in the interval $0 < x < 1/2$ (with the first zero). Recall that we then have

$$g_\infty(x) = \frac{1}{2x^{3/2}}, \quad 1/2 < x < 1,$$

and

$$g_\infty(x) = \pi(\frac{s_0}{x^{3/2}} + \frac{2(-s_0)^2}{x^{5/2}}) \exp(s_0(1 + \frac{1}{x})), \quad 0 < x < 1/2.$$

Using these densities, we can get an estimate for any moment by integration. Especially, we have

$$\mathbf{E}(V_1) = 0.4833253830, \quad \mathbf{Var}(V_1) = 0.0495564822.$$

We see that the first three decimals equals the expectation and variance given by the asymptotic formulas above. Below, we tabulate the first four moments.

Moment	
1:st	0.483
2:nd	0.283
3:rd	0.190
4:th	0.140

Table 6: First four moments for longest excursion

5 2:nd and 3:rd longest excursion of the Brownian bridge

A very good approximation of the asymptotic properties of the n:th longest excursion is apprehendable by using the methods of Section 4.3. The representation (11) of the longest excursion V_1 works very well for small x. Then we realise that the approximations of V_n for n greater than 2 will probably work even better (because $V_i \geq V_j$ for $i < j$). First, we consider Kummer representations for the 2:nd and 3:rd longest excursions. Then, we study some of their interesting properties. The distribution for the n:th longest excursion can be attained in a similar way as for the 2:nd and 3:rd.

5.1 Proof of a Kummer representation of the second longest excursion

An analytic expression for the distribution of the n:th longest excursion can be attained by using Remark 4.3.1. Here, we present a result for the 2:nd longest excursion.

Corollary 5.1.1

$$\begin{aligned}
\mathbf{P}(V_2 \leq x) &= 2\mathbf{P}(V_1 \leq x) - \frac{\pi^{3/2}}{\sqrt{x}}\Bigg[4\sum_{k\neq 0} s_k^2 e^{s_k t}\Big((t-1)\sqrt{s_k} + \frac{1}{\sqrt{s_k}}\Big) \\
&- \frac{1}{\pi}\int_0^\infty e^{-vt}\sqrt{v}H(-v)dv + \frac{Ae^{s_0 t}}{\sqrt{\pi}}\Big((1+s_0 t)\int_0^t \frac{e^{-s_0\tau}}{\sqrt{\tau}}d\tau \\
&- s_0\int_0^t \sqrt{\tau}e^{-s_0\tau}d\tau\Big) + B\Big(\frac{1}{\sqrt{\pi t}} + \frac{s_0 e^{s_0 t}}{\sqrt{\pi}}\int_0^t \frac{e^{-s_0\tau}}{\sqrt{\tau}}d\tau\Big)\Bigg],
\end{aligned}$$

where

$$A = 4{s_0}^2, \quad B = 4{s_0}^2(\frac{1}{s_0} - 1), \quad t = \frac{1}{x} + 2,$$

and

$$H(s) = \frac{1}{\Phi(1,\frac{1}{2},s)^2} - \frac{A}{(s-s_0)^2} - \frac{B}{s-s_0},$$

and s_k are the zeros in the Kummer function $\Phi(1,\frac{1}{2},s)$.

Proof. Remark 4.3.1 and Theorem 4.3.1 give

$$\begin{aligned}
&\mathbf{P}(V_2 \leq x) = \mathbf{P}(M \leq 1) = \mathbf{P}(M=0) + \mathbf{P}(M=1) \\
=\ & \mathbf{P}(V_1 \leq x) + \frac{1}{2\sqrt{x}i}\int_{-i\infty}^{i\infty} e^{s(\frac{1}{x}+1)}\frac{(\Phi(1,\frac{1}{2},s) - e^s\sqrt{\pi s})}{(\Phi(1,\frac{1}{2},s))^2}ds \\
=\ & \mathbf{P}(V_1 \leq x) + \frac{1}{2\sqrt{x}i}\Big(\int_{-i\infty}^{i\infty}\frac{e^{s(\frac{1}{x}+1)}}{\Phi(1,\frac{1}{2},s)}ds - \int_{-i\infty}^{i\infty}\frac{e^{s(\frac{1}{x}+2)}\sqrt{\pi s}}{(\Phi(1,\frac{1}{2},s))^2}ds\Big) \\
=\ & 2\mathbf{P}(V_1 \leq x) - \frac{\sqrt{\pi}}{2\sqrt{x}i}\int_{-i\infty}^{i\infty}\frac{e^{s(\frac{1}{x}+2)}\sqrt{s}}{(\Phi(1,\frac{1}{2},s))^2}ds.
\end{aligned}$$

Let $t = \frac{1}{x} + 2$ and consider

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{st}\sqrt{s}}{(\Phi(1,\frac{1}{2},s))^2}ds.$$

We want to calculate this by residue calculus, but are annoyed by the pole at $s = s_0$, because of the principal branch of $\sqrt{s}$. This problem is solved by

$$\begin{aligned}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{st}\sqrt{s}}{(\Phi(1,\frac{1}{2},s))^2}ds &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\sqrt{s}H(s)ds \\ + \quad \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{Ae^{st}\sqrt{s}}{(s-s_0)^2}ds &+ \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{Be^{st}\sqrt{s}}{s-s_0}ds,\end{aligned}$$

where

$$\frac{A}{(s-s_0)^2} \text{ is appropriate term in the Laurent series around } s_0,$$

$$\frac{B}{s-s_0} \text{ is appropriate term in the Laurent series around } s_0,$$

$$H(s) = \frac{1}{\Phi(1,\frac{1}{2},s)^2} - \frac{A}{(s-s_0)^2} - \frac{B}{s-s_0}.$$

Then,

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{st}\sqrt{s}}{(s-s_0)^2}ds = \mathcal{L}^{-1}\{\frac{\sqrt{s}}{(s-s_0)^2}\} \\ &= \mathcal{L}^{-1}\{\frac{1}{\sqrt{s}(s-s_0)} + \frac{s_0}{\sqrt{s}(s-s_0)^2}\} = \frac{1}{\sqrt{\pi t}} * e^{s_0 t} + s_0\left(\frac{1}{\sqrt{\pi t}} * (te^{s_0 t})\right) \\ &= \frac{e^{s_0 t}}{\sqrt{\pi}}(1+s_0 t)\int_0^t \frac{e^{-s_0\tau}}{\sqrt{\tau}}d\tau - \frac{e^{s_0 t}}{\sqrt{\pi}}s_0\int_0^t \sqrt{\tau}e^{-s_0\tau}d\tau,\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{st}\sqrt{s}}{(s-s_0)}ds = \mathcal{L}^{-1}\{\frac{\sqrt{s}}{(s-s_0)}\} = \mathcal{L}^{-1}\{\frac{1}{\sqrt{s}} + \frac{s_0}{\sqrt{s}(s-s_0)}\} \\ &= \frac{1}{\sqrt{\pi t}} + s_0\left(\frac{1}{\sqrt{\pi t}} * (e^{s_0 t})\right) = \frac{1}{\sqrt{\pi t}} + \frac{e^{s_0 t}}{\sqrt{\pi}}s_0 t\int_0^t \frac{e^{-s_0\tau}}{\sqrt{\tau}}d\tau.\end{aligned}$$

Finally, we have to calculate

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\sqrt{s}H(s)ds. \tag{18}$$

We have constructed $H(s)$ so that it is analytic everywhere except for the poles s_k ($k \neq 0$). We now solve (18) by residue calculus:

$$
\begin{aligned}
&\lim_{A\to\infty}\left(\frac{1}{2\pi i}\int_{\Gamma_1} e^{st}\sqrt{s}H(s)ds+\frac{1}{2\pi i}\int_{\Gamma_2} e^{st}\sqrt{s}H(s)ds\right.\\
&+\frac{1}{2\pi i}\int_{\Gamma_3} e^{st}\sqrt{s}H(s)ds+\frac{1}{2\pi i}\int_{\Gamma_4} e^{st}\sqrt{s}H(s)ds\\
&\left.+\frac{1}{2\pi i}\int_{\Gamma_5} e^{st}\sqrt{s}H(s)ds\right)=\sum_{k\neq 0}\mathrm{Res}\{H(s);s_k\},
\end{aligned}
$$

where $\Gamma_1=\{-iA<s<iA\}$, $\Gamma_2=\{s=Ae^{i\theta}:\frac{\pi}{2}<\theta<\pi\}$, $\Gamma_3=\{s=Ae^{i\theta}:\pi<\theta<\frac{3\pi}{2}\}$, $\Gamma_4=\{-A<s<A\}$ and $\Gamma_5=\{-A>s>A\}$.

But

$$
\begin{aligned}
&\lim_{A\to\infty}\left(\frac{1}{2\pi i}\int_{\Gamma_4} e^{st}\sqrt{s}H(s)ds+\frac{1}{2\pi i}\int_{\Gamma_5} e^{st}\sqrt{s}H(s)ds\right)\\
&=\frac{1}{\pi}\int_0^\infty e^{-xt}\sqrt{x}H(-x)dx.
\end{aligned}
$$

Also

$$
\lim_{A\to\infty}\left(\frac{1}{2\pi i}\int_{\Gamma_1} e^{st}\sqrt{s}H(s)ds\right)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\sqrt{s}H(s)ds,
$$

and

$$
\lim_{A\to\infty}\left(\frac{1}{2\pi i}\int_{\Gamma_2} e^{st}\sqrt{s}H(s)ds+\frac{1}{2\pi i}\int_{\Gamma_3} e^{st}\sqrt{s}H(s)ds\right)=0.
$$

This follows in the same way as for the longest excursion.

At last, we have

$$
\mathrm{Res}\{H(s);s_k\}=4{s_k}^2e^{s_kt}\left((t-1)\sqrt{s_k}+\frac{1}{\sqrt{s_k}}\right), \tag{19}
$$

and

$$
A=4{s_0}^2,\quad B=4{s_0}^2(\frac{1}{s_0}-1).
$$

Equation (19) follows from extracting the correct term in the Laurent series of $H(s)$ around $s=s_k$, and the last two expressions follows from extracting the correct terms in the Laurent series of $\frac{1}{\Phi(1,\frac{1}{2},s)^2}$ around $s=s_0$.

Putting everything together, we get the assertion.

□

5.2 Study of the Kummer representation of the second longest excursion

Corollary 5.1.1 gives a way to study the distribution and density of the second longest excursion. The expressions in Corollary 5.1.1 are half-analytic, meaning that they need to be computed numerically in some way. In particular, the integrals in Corollary 5.1.1 need to be numerically computed. This has to be done using a program as MAPLE or MATLAB. We do this, using MAPLE, and find

$$\mathbf{P}(V_2 < \frac{1}{2}) \approx 0.99957,$$

only including the first three zeros of the Kummer function. We know that this should be 1, so the error is small. We also compute some further values to get the distribution and density, see figures 15 and 16. Note that the density is apprehended by numerical derivation of the distribution.

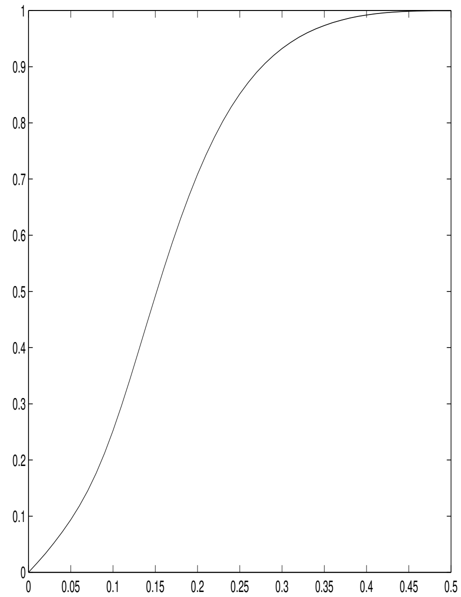

Figure 15: Distribution of the second longest excursion.

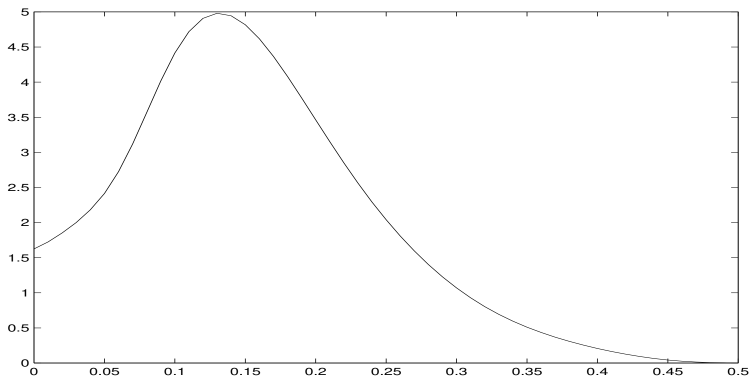

Figure 16: Density of the second longest excursion.

Note that there is a positive density that the excursion will be of length zero. This happens when the longest excursion is of length 1 (which has a positive density) or when the trajectory 'oscillates all the time' (which has probability zero).

5.3 Proof of a Kummer representation of the third longest excursion

The 3:rd longest excursion is given by

Corollary 5.3.1

$$\begin{aligned}\mathbf{P}(V_3 \leq x) = \quad & 3\Big(\mathbf{P}(V_2 \leq x) - \mathbf{P}(V_1 \leq x)\Big) + \\ & + \frac{2\pi^2}{\sqrt{x}} \sum_k {s_k}^2 e^{s_k t} \Big((-4 + 6t - 2t^2){s_k}^2 + (12 - 7t)s_k - 3 \Big),\end{aligned}$$

where $t = \frac{1}{x} + 3$ *and* s_k *are the zeros in the Kummer function* $\Phi(1, \frac{1}{2}, s)$.

Proof. Remark 4.3.1 and Theorem 4.3.1 give

$$\begin{aligned}\mathbf{P}(V_3 \leq x) &= \mathbf{P}(M \leq 1) = \mathbf{P}(M = 0) + \mathbf{P}(M = 1) + \mathbf{P}(M = 2) \\ &= 3\Big(\mathbf{P}(V_2 \leq x) - \mathbf{P}(V_1 \leq x)\Big) + \frac{\pi}{2\sqrt{x}i} \int_{-i\infty}^{i\infty} \frac{e^{s(\frac{1}{x}+3)} s}{(\Phi(1, \frac{1}{2}, s))^3} ds.\end{aligned}$$

Now, let $t = \frac{1}{x} + 3$ and consider

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{st} s}{(\Phi(1, \frac{1}{2}, s))^3} ds.$$

The Residue Theorem gives

$$\lim_{A\to\infty}\left(\frac{1}{2\pi i}\int_{\Gamma_1}\frac{e^{st}s}{(\Phi(1,\frac{1}{2},s))^3}ds+\frac{1}{2\pi i}\int_{\Gamma_2}\frac{e^{st}s}{(\Phi(1,\frac{1}{2},s))^3}ds\right)=$$
$$=\sum_k \mathrm{Res}\{\frac{e^{st}s}{(\Phi(1,\frac{1}{2},s))^3};s_k\},$$

where $\Gamma_1=\{-iA<s<iA\}$ and $\Gamma_2=\{s=Ae^{i\theta}:\frac{\pi}{2}<\theta<\frac{3\pi}{2}\}$.

If we expand $\frac{e^{st}s}{(\Phi(1,\frac{1}{2},s))^3}$ in a Laurent series around s_k, we get

$$\mathrm{Res}\{\frac{e^{st}s}{(\Phi(1,\frac{1}{2},s))^3};s_k\}=2{s_k}^2e^{s_kt}\left((-4+6t-2t^2){s_k}^2+(12-7t)s_k-3\right).$$

We also have

$$\lim_{A\to\infty}\left(\frac{1}{2\pi i}\int_{\Gamma_2}\frac{e^{st}s}{(\Phi(1,\frac{1}{2},s))^3}ds\right)=0.$$

This follows in the same way as for the longest excursion.

The Corollary is thus proved.

□

5.4 Study of the Kummer representation of the third longest excursion

Corollary 5.3.1 gives the distribution and density of the second longest excursion. In the same way as for the 2:nd longest excursion, we can use MAPLE to compute the distribution of the 3:rd longest excursion. For example, we get

$$\mathbf{P}(V_3<0.33)\approx 1.00014,$$

only using the first zero of the Kummer function. We know that this should be close to 1 (exactly 1 for $x=1/3$), so the error is small. The distribution and density are plotted in figures 17 and 18.

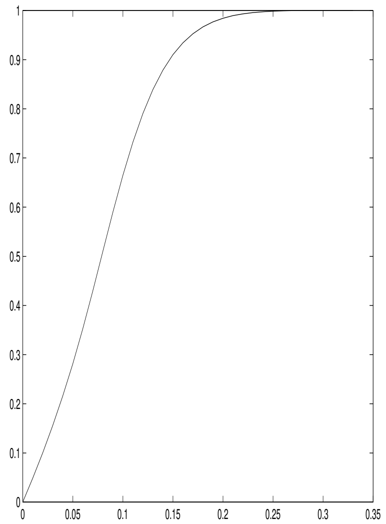

Figure 17: Distribution of the third longest excursion.

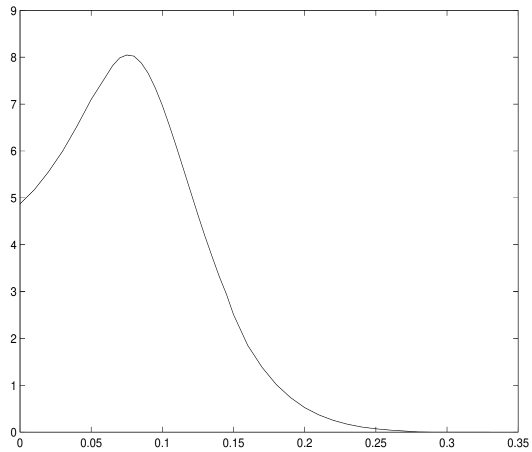

Figure 18: Density of the third longest excursion.

5.5 Comparison of the distributions

We have representations for the distributions of the three longest excursions. The distributions are plotted in figure 19 and the densities are plotted in figure 20. Note that the plotted distributions are the truncated Kummer representations.

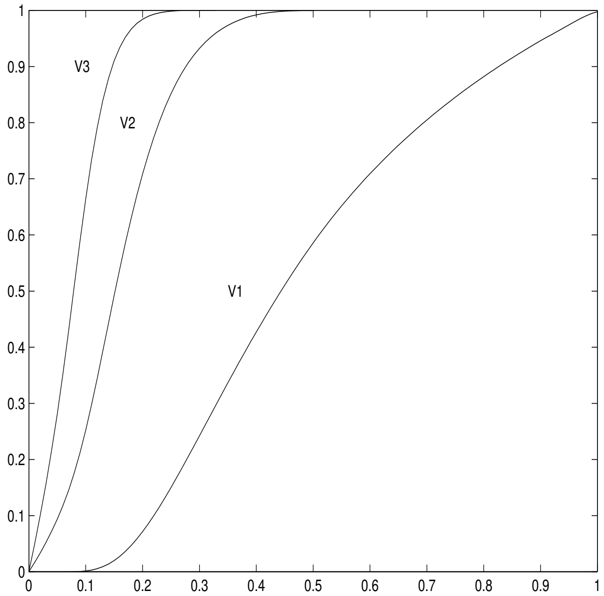

Figure 19: Distributions of the three longest excursions.

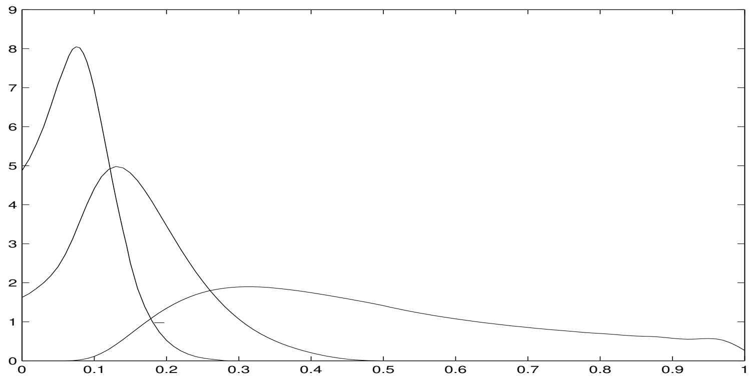

Figure 20: Densities of the three longest excursions.

To get a better grasp of the behaviour near zero, we plot $\mathbf{P}(V_i \leq \frac{1}{x}) = \mathbf{P}(x \leq \frac{1}{V_i})$.

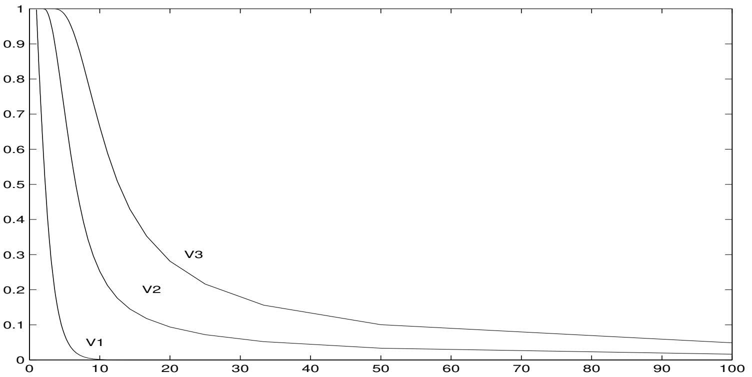

Figure 21: $\mathbf{P}(V_i \leq \frac{1}{x}), i = 1, 2, 3.$

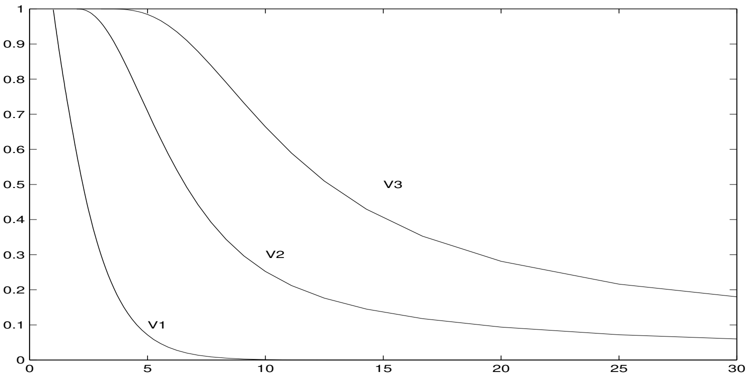

Figure 22: $\mathbf{P}(V_i \leq \frac{1}{x}), i = 1, 2, 3.$

We can see that $\mathbf{P}(V_1 \leq \frac{1}{x})$ decays rapidly to zero which indicates that the density of the longest excursion is zero at $x = 0$. The slow decay to zero for $\mathbf{P}(V_2 \leq \frac{1}{x})$ and $\mathbf{P}(V_3 \leq \frac{1}{x})$ indicates a positive density for the 2:nd and 3:rd longest excursions at $x = 0$.

Remark 4.3.1 states that $\mathbf{P}(V_j \le x) = \sum_{k=0}^{j-1} \mathbf{P}(M = k)$, where $M = \sum_{j=1}^{\infty} I(V_j > x)$. This can be illustrated in a figure involving the distributions for the j longest excursions. Figure 23 illustrates this for the example $\mathbf{P}(V_3 \le x) = \mathbf{P}(M = 0) + \mathbf{P}(M = 1) + \mathbf{P}(M = 2)$. We can interpret the part $\mathbf{P}(M = j)$ as the probability that the j longest excursions are greater than x and the rest are smaller than x for some chosen x.

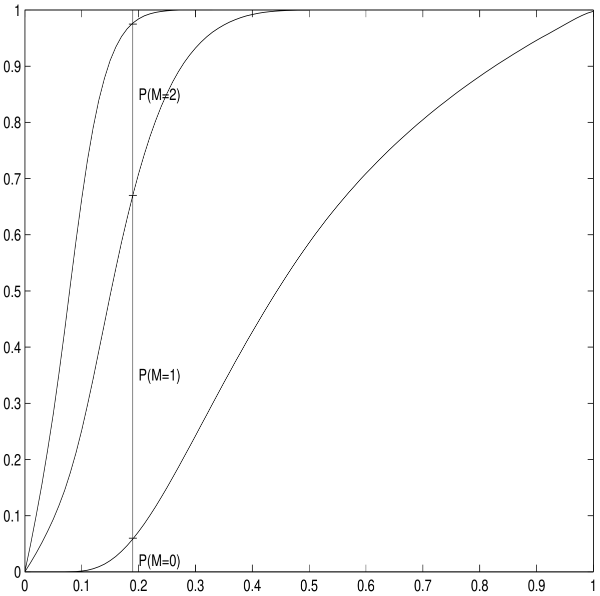

Figure 23: Illustration of $\mathbf{P}(V_3 \le x) = \mathbf{P}(M = 0) + \mathbf{P}(M = 1) + \mathbf{P}(M = 2)$.

6 Weak convergence of the distribution of excursions

In order for the results in Sections 4 and 5 to be complete, we have to show that the functional that picks out the n:th longest excursion is continuous. Then, we can use the functional central limit theorem to show that this functional in the suitably normalized and scaled tied down simple random walk converges weakly to the corresponding functional in the Brownian bridge. Also, we have to show that the conditioning on $\{S_{2n} = 0\}$ converges in the right sense so that we get a Brownian bridge. These issues are left out in this paper.

7 Concluding remarks

This paper explores the distribution of the longest excursion in a tied down simple random walk and the longest excursions in the Brownian bridge. There is a representation that actually gives the distribution of the longest excursion for any given tied down random walk. This result is easily implemented using MAPLE. The next step is to study the behaviour when the suitably scaled and normalized random walk is taken to the limit. The main tool for acchieving this in this paper is by studying the Laplace transform of a sum of indicators, indicating that the n:th excursion is greater than a specified x. The Laplace transform can be inverted in general and gives an integral representation involving a Kummer function. The integral can be evaluated, using calculus of residues. This gives the distribution for the n:th longest excursion. These expressions involve infinite sums containing the zeros of the Kummer function. Truncating for only the first zero, gives a simple formula, which gives very accurate approximations. This is explicitly done for the three longest excursions. The closed expressions are half-analytic, meaning that they need to be evaluated numerically. Closed expressions for the longest excursion have been obtained previously in different versions, but then only for certain intervals ranging from 1 and downwards to $1/4$. The Kummer representation, on the other hand gives a very good approximation for values of x smaller than say $1/2$ (truncated at the first zero of the Kummer function). Combining these two results, one has a good grasp of the whole distribution of the longest excursion. Also the Kummer representation gives a very good approximation for the whole distribution of the n:th longest excursion (truncated at the first zero of the Kummer function).

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A Properties of the Kummer function

Below, we tabulate the first 21 zeros of the Kummer function:

k	$a_{\pm k}$	b_k	b_{-k}
0	0.8540326566	0	0
1	4.248920779	6.383124294	-6.383124294
2	5.184114730	12.88530517	-12.88530517
3	5.752397135	19.27404104	-19.27404104
4	6.163099738	25.62068932	-25.62068932
5	6.485136309	31.94674125	-31.94674125
6	6.750133543	38.26105030	-38.26105030
7	6.975305915	44.56797780	-44.56797780
8	7.171079275	50.86993830	-50.86993830
9	7.344252459	57.16838389	-57.16838389
10	7.499507360	63.46424369	63.46424369

Table 7: Zeros of the Kummer function. $s_k = -a_k + b_k i$.

The zeros are apprehended by MAPLE.

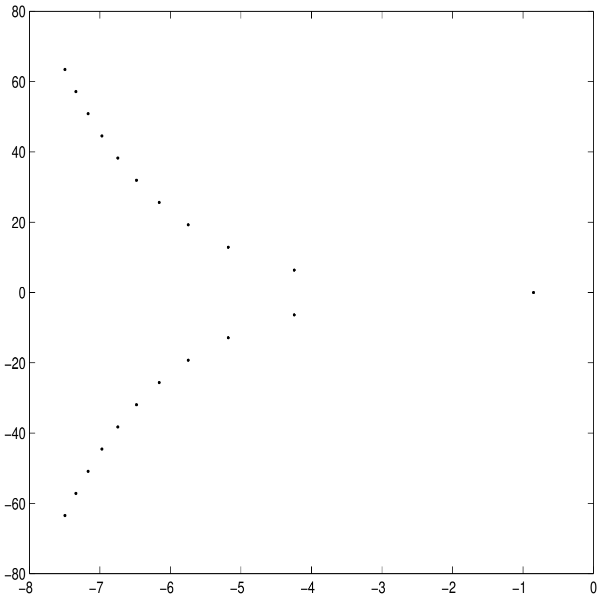

Figure 24: The first 21 zeros plotted in the complex plane.

According to Wendel ([11]), $s_k = -a_k + b_k i$ where k is an integer and

$$a_k \sim 3\ln(k), \quad b_k \sim 2\pi k, \quad k \to \infty.$$

The derivatives of the Kummer function can be expressed with the Kummer function itself as

$$\Phi'(1, \frac{1}{2}, s) = \Phi(1, \frac{1}{2}, s)(1 + \frac{1}{2s}) - \frac{1}{2s},$$

$$\Phi''(1, \frac{1}{2}, s) = \Phi'(1, \frac{1}{2}, s)(1 + \frac{1}{2s}) - \Phi(1, \frac{1}{2}, s)\frac{1}{2s^2} + \frac{1}{2s^2},$$

$$\Phi'''(1, \frac{1}{2}, s) = \Phi''(1, \frac{1}{2}, s)(1 + \frac{1}{2s}) - \Phi'(1, \frac{1}{2}, s)\frac{1}{s^2} + \Phi(1, \frac{1}{2}, s)\frac{1}{s^3} - \frac{1}{s^3}.$$

B Numerical inversion of Laplace transforms

[1], Abate and Whitt 1995, presents two ways of numerical inversion of Laplace transforms. Method one in [1], implemented in Matlab, gives the approximate inversion in figure 25 for the integral

$$\frac{1}{2\sqrt{x}i}\int_{-i\infty}^{i\infty} e^{s(1+1/x)}\frac{(\Phi(1,\frac{1}{2},s)-e^{s}\sqrt{\pi s})^{j}}{(\Phi(1,\frac{1}{2},s))^{j+1}}ds.$$

This should be compared with the true distribution. Figure 25 shows the disappointing result that the inversion does not seem to work.

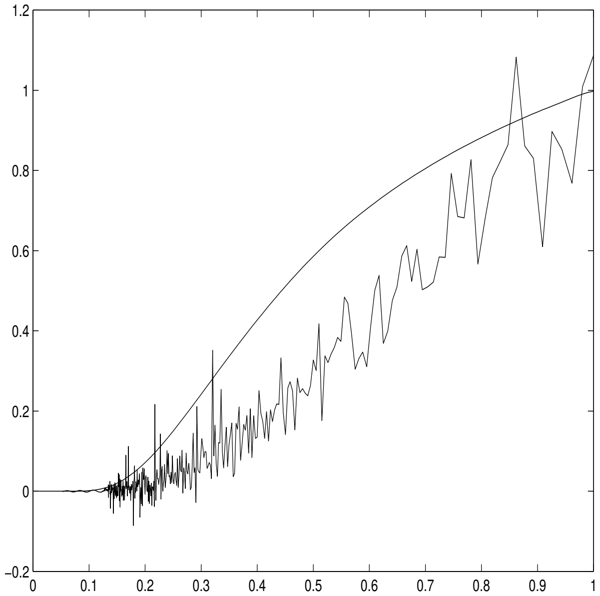

Figure 25: Numerically inverted distribution vs true distribution.

Method one in [1] solves the following real integral numerically with the Trapezoid method and Euler summation

$$f(t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\hat{f(s)}ds=\ldots=\frac{2}{\pi}\int_{0}^{\infty}\mathrm{Re}(\hat{f(iu)})\cos{(ut)}du.$$

C Computer program for numerical evaluation of recursive relation

The computer program is intented for MATLAB.

```
clear;
Nx=100;
Nu=100;
stepx=1/Nx;
ustep=1/(2*Nu);
G(1)=1;

for i=1:Nx
 x=1-(i-1)*stepx;
 xdelta=1-i*stepx;
 X(i)=x;
 uint=0;

 for j=1:Nu
  if ((1-x)==0)
   Gintfirst=1;
   Gintsec=1;
  else
   xval1=x/((1-x)*(j*ustep));
   xval2=x/((1-x)*(1-j*ustep));
   for k=1:i
    if (xval1>1-((i+1-k)-1)*stepx)
     Gintfirst=G(i+1-k);
    end
    if (xval2>1-((i+1-k)-1)*stepx)
     Gintsec=G(i+1-k);
    end
   end;
   if (xval1>1)
    Gintfirst=1;
   end
   if (xval2>1)
    Gintsec=1;
   end
  end
  uint=uint+Gintfirst*Gintsec*ustep/
(pi*sqrt(j*ustep*(1-j*ustep)));
 end;
 Gnew=G(i)-uint*stepx/(x^(3/2));
```

```
  G(i+1)=Gnew;
end;

G=G';
X(i+1)=0;
X=X';

lg=length(G);
for i=1:(lg-1)
 dens(i)=G(i)-G(i+1);
end
dens(lg)=G(lg)-0;
dens=Nx*dens;

subplot(1,2,1)
plot(X,G);
title('Numerical Asymptotic Distribution')

subplot(1,2,2)
plot(X,dens);
title('Numerical Asymptotic Density')
```