FIRST PASSAGE TIME FOR RANDOM WALKS AND BIRTH-AND-DEATH PROCESSES WITH SIGN DEPENDING TRANSITION PROBABILITIES

Anna Carlsund

Abstract

Consider a random walk with sign depending transition probabilities. Let it start in 0 and stop the random walk when it reaches an upper border, $a > 0$, or a lower border, $-b < 0$, for the first time. This paper investigates when and where the process stops. If we let the holding times be independent exponentially distributed with an intensity depending on the sign of the state, when and where does this process stop? When the time and state space are appropriately scaled we can deduce a limit process. Some extensions of the first passage time are derived. All distributions are derived through transforms; probability generating functions are used in the discrete cases and Laplace transforms are used in the continuous cases. The transforms are so complicated that it is not feasible to invert them analytically. With a mathematical computer program it is possible to get the Taylor expansion from the probability generating function and therefore also the distribution. The Laplace transforms are numerically inverted.

SIGN DEPENDING TRANSITION PROBABILITIES; SIMPLE RANDOM WALK; CONTINUOUS TIME MARKOV CHAIN; FIRST PASSAGE TIME; LAPLACE TRANSFORMS; INVERSION OF LAPLACE TRANSFORMS; COVER TIME; RANGE.

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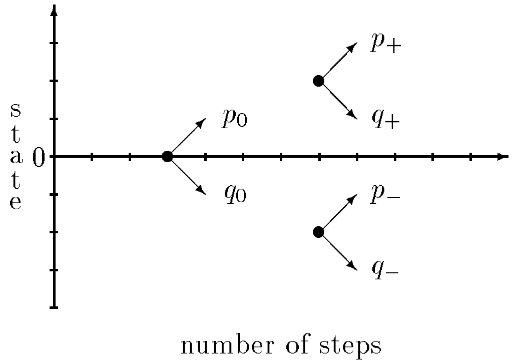


Figure 1: The transition probabilities for the random walk $\{S_n\}_{n \geq 0}$.

1 Introduction

Consider a random walk, $\{S_n\}_{n \geq 0}$, with the transition probabilities depending on the state. Partition the state space in three parts, above 0, below 0 and state 0. If S_n is in a positive state i, let the probability of jump from state i to state $i+1$ be p_+, similarly if S_n is in a negative state let the probability of jump one step up be p_- and if S_n is in state 0 let the probability of jump to state 1 be p_0, see Figure 1.

Set an upper border, $a > 0$, and a lower border, $-b < 0$. Let $\{S_n\}_{n \geq 0}$ start in 0 and stop the random walk when it reaches one of the borders for the first time. In Section 2 we investigate how many steps it takes until the random walk is stopped, and with which probability it stops in a respectively $-b$. In Section 3 the holding times are independent exponentially distributed where the expected holding times depend on the state in a similar way as the transition probabilities. In Section 4, the length of the jumps and the time between the jumps are scaled and when they tend to 0 a limit distribution is obtained. In Section 5 some extensions of the first passage time are derived.

2 First passage time for a random walk with sign depending transition probabilities

Let N be the minimum number of steps before the random walk reaches one of the borders the first time, i.e.

$$N \stackrel{\text{def}}{=} \min\{n : S_n = a \text{ or } S_n = -b\}. \tag{1}$$

A random walk which starts in 0 and is stopped the first time it hits a border

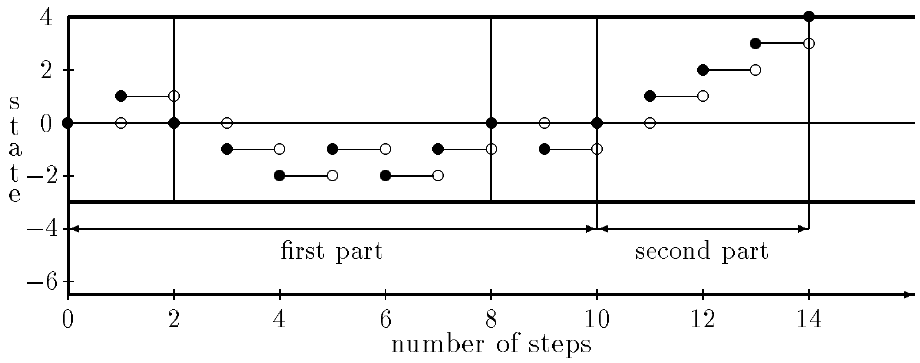


Figure 2: A realization of the random walk $\{S_n\}_{n\geq 0}$ with borders $a=4$ and $b=3$.

has a certain behaviour. First it starts in 0, then it returns to 0 a random number of times without reaching any of the borders and then finally it reaches one of the borders and the random walk is stopped. Thus the random walk can be divided into two independent parts, from the start to the last time the random walk visits 0 and from the last visit to 0 until it is absorbed in one of the borders, see Figure 2.

Considering the first part, it can be split by the returns to 0, see Figure 2. The number of steps between two successive visits to 0 is random. Let the random variable Z_i denote the number of steps between the *i-1*th and the i^{th} return to 0, $Z_0 \equiv 0$, and let N_0 be the number of times the random walk returns to 0. Thus the first part of the random walk can be written as

$$\sum_{i=0}^{N_0} Z_i. \tag{2}$$

For an example see Figure 3, where the first part is $2+6+2=10$ steps.

Considering the second part, define Z_{abs} to be the number of steps from the last visit to 0 until absorption, see Figure 3 where $Z_{abs}=4$.

Adding the first and the second parts the total number of steps before absorption, N, can be written as

$$N = \sum_{i=0}^{N_0} Z_i + Z_{abs}.$$

Using that the number of steps in the first and the second part are independent, the probability generating function of N fulfills

$$E\left[x^N\right] = E\left[x^{\sum_{i=0}^{N_0} Z_i + Z_{abs}}\right] = E\left[x^{\sum_{i=0}^{N_0} Z_i}\right] E\left[x^{Z_{abs}}\right]. \tag{3}$$

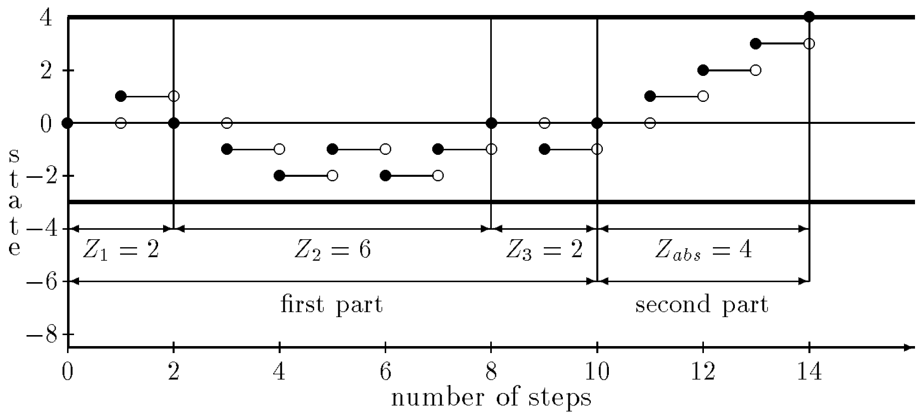


Figure 3: The parameters used are: $p_0 = p_+ = p_- = 0.5$, $a = 4$ and $b = 3$. Here $N_0 = 3$ and $N = 14$.

The first factor is considered in Section 2.1 and the second factor in Section 2.2.

2.1 Calculation of $E\left[x^{\sum_{i=0}^{N_0} Z_i}\right]$

For a random walk starting at x introduce the random variables

$$\tau_x^y = \min\{n \geq 1 : S_n = y\}, \tag{4}$$

$$\tau_x^{y,z} = \min(\tau_x^y, \tau_x^z) \tag{5}$$

and set

$$p_{abs} = P(\tau_0^{a,-b} < \tau_0^0), \tag{6}$$

$$q_{abs} = P(\tau_0^0 < \tau_0^{a,-b}). \tag{7}$$

To derive the probability generating function of N it is not necessary to get an explicit expression for (6) and (7), but they are used later to derive $P(S_N = a)$. Conditioning on the first step in the random walk and using the formula in Theorem 1 in [4], we get

$$p_{abs} = p_0 e^{(a-1)\zeta_+} \frac{\mathrm{sh}(\zeta_+)}{\mathrm{sh}(a\zeta_+)} + q_0 e^{-(b-1)\zeta_-} \frac{\mathrm{sh}(\zeta_-)}{\mathrm{sh}(b\zeta_-)}$$

$$q_{abs} = p_0 e^{-\zeta_+} \frac{\mathrm{sh}((a-1)\zeta_+)}{\mathrm{sh}(a\zeta_+)} + q_0 e^{\zeta_-} \frac{\mathrm{sh}((b-1)\zeta_-)}{\mathrm{sh}(b\zeta_-)}$$

where $\mathrm{sh}(\cdot) = \sinh(\cdot)$ and $\zeta_\pm = \ln\sqrt{\frac{p_\pm}{q_\pm}}$.

As the Z_i's are independent identically distributed ($Z_0 \equiv 0$) and independent of N_0 we have

$$E\Big[x^{\sum_{i=0}^{N_0} Z_i}\Big] = E\Big[E\Big[x^{\sum_{i=0}^{N_0} Z_i}|N_0\Big]\Big] = E\Big[E\big[x^{Z_1}\big]^{N_0}\Big]. \tag{8}$$

Every time the random walk is in state 0, it hits a border before hitting 0 with probability p_{abs} and vice versa with probability q_{abs}. Every try to reach a border is independent of the previous tries. This implies that N_0, the number of returns to 0, is geometrically distributed with parameter p_{abs}, i.e.

$$P(N_0 = n) = q_{abs}^n p_{abs}, \qquad n \geq 0, \tag{9}$$

and therefore

$$E\Big[E\big[x^{Z_1}\big]^{N_0}\Big] = \frac{p_{abs}}{1 - q_{abs}E[x^{Z_1}]}. \tag{10}$$

Now

$$E\big[x^{Z_1}\big] = \sum_{k=2}^{\infty} x^k P(Z_1 = k) = \sum_{k=2}^{\infty} x^k P(\tau_0^0 = k|\tau_0^{a,-b} > \tau_0^0)$$

$$= \sum_{k=2}^{\infty} x^k \frac{P(\tau_0^0 = k, \tau_0^{a,-b} > \tau_0^0)}{P(\tau_0^{a,-b} > \tau_0^0)} = \frac{1}{q_{abs}} \sum_{k=2}^{\infty} x^k P(\tau_0^0 = k, \tau_0^{a,-b} > \tau_0^0),$$

and conditioning on the first step of the random walk

$$\begin{aligned} E\big[x^{Z_1}\big] = \frac{1}{q_{abs}} \sum_{k=2}^{\infty} x^k \Big(& P(\tau_0^0 = k, \tau_0^{a,-b} > \tau_0^0|X_1 = 1)p_0 + \\ & P(\tau_0^0 = k, \tau_0^{a,-b} > \tau_0^0|X_1 = -1)q_0\Big) \\ = \frac{x}{q_{abs}} \sum_{k=1}^{\infty} x^k \Big(& P(\tau_1^0 = k, \tau_1^a > \tau_1^0)p_0 + P(\tau_{-1}^0 = k, \tau_{-1}^{-b} > \tau_{-1}^0)q_0\Big). \end{aligned}$$

The first part of this sum, except for $\frac{xp_0}{q_{abs}}$, is the probability generating function for the time it takes a simple random walk, which starts in 1 and have transition probabilities p_+ everywhere, to reach 0 before a. This is the same probability generating function as that of a simple random walk with start in 0 and borders -1 and $a-1$, see Theorem 1 in [4]. Analogously the second part of the sum can be considered.

Thus with the notation used in [4],

$$E\left[x^{Z_1}\right] = \frac{x}{q_{abs}}(U_1^+(x)p_0 + U_{-1}^-(x)q_0)$$

$$= \frac{x\left(p_0 e^{-\zeta_+}\frac{\operatorname{sh}((a-1)\eta_+(x))}{\operatorname{sh}(a\eta_+(x))} + q_0 e^{\zeta_-}\frac{\operatorname{sh}((b-1)\eta_-(x))}{\operatorname{sh}(b\eta_-(x))}\right)}{q_{abs}} \tag{11}$$

where $\eta_\pm(x) = \operatorname{arccosh}\left(\frac{1}{x\sqrt{4p_\pm q_\pm}}\right)$. Substituting (11) into (10) gives

$$E\left[x^{\sum_{i=0}^{N_0} Z_i}\right] = \frac{p_{abs}}{1 - x\left(p_0 e^{-\zeta_+}\frac{\operatorname{sh}((a-1)\eta_+(x))}{\operatorname{sh}(a\eta_+(x))} + q_0 e^{\zeta_-}\frac{\operatorname{sh}((b-1)\eta_-(x))}{\operatorname{sh}(b\eta_-(x))}\right)}. \tag{12}$$

2.2 Calculation of $E\left[x^{Z_{abs}}\right]$

The probability generating function of Z_{abs} is derived in a similar way as $E\left[x^{Z_1}\right]$. By definition Z_{abs} is conditioned on being absorbed in either a or $-b$ without passing 0. Split $E\left[x^{Z_{abs}}\right]$ into two cases:

$$E\left[x^{Z_{abs}}\right] = E\left[x^{Z_{abs}} I_{S_N=a}\right] + E\left[x^{Z_{abs}} I_{S_N=-b}\right].$$

Conditioning on the first step and using Theorem 1 in [4], we get

$$E\left[x^{Z_{abs}} I_{S_N=a}\right] = \frac{p_0 x e^{(a-1)\zeta_+}\frac{\operatorname{sh}(\eta_+(x))}{\operatorname{sh}(a\eta_+(x))}}{p_{abs}}, \tag{13}$$

$$E\left[x^{Z_{abs}} I_{S_N=-b}\right] = \frac{q_0 x e^{-(b-1)\zeta_-}\frac{\operatorname{sh}(\eta_-(x))}{\operatorname{sh}(b\eta_-(x))}}{p_{abs}},$$

and

$$E\left[x^{Z_{abs}}\right] = \frac{x\left(p_0 e^{(a-1)\zeta_+}\frac{\operatorname{sh}(\eta_+(x))}{\operatorname{sh}(a\eta_+(x))} + q_0 e^{-(b-1)\zeta_-}\frac{\operatorname{sh}(\eta_-(x))}{\operatorname{sh}(b\eta_-(x))}\right)}{p_{abs}}. \tag{14}$$

2.3 The probability generating function of N

Substituting (12) and (14) in (3) we get

$$E\Big[x^N\Big] = p_{abs}\frac{x\left(\frac{p_0\, e^{(a-1)\zeta_+}\frac{\mathrm{sh}(\eta_+(x))}{\mathrm{sh}(a\eta_+(x))}+q_0\, e^{-(b-1)\zeta_-}\frac{\mathrm{sh}(\eta_-(x))}{\mathrm{sh}(b\eta_-(x))}}{p_{abs}}\right)}{1-x\left(p_0 e^{-\zeta_+}\frac{\mathrm{sh}((a-1)\eta_+(x))}{\mathrm{sh}(a\eta_+(x))}+q_0 e^{\zeta_-}\frac{\mathrm{sh}((b-1)\eta_-(x))}{\mathrm{sh}(b\eta_-(x))}\right)}$$

$$= \frac{x\left(p_0 e^{(a-1)\zeta_+}\frac{\mathrm{sh}(\eta_+(x))}{\mathrm{sh}(a\eta_+(x))}+q_0 e^{-(b-1)\zeta_-}\frac{\mathrm{sh}(\eta_-(x))}{\mathrm{sh}(b\eta_-(x))}\right)}{1-x\left(p_0 e^{-\zeta_+}\frac{\mathrm{sh}((a-1)\eta_+(x))}{\mathrm{sh}(a\eta_+(x))}+q_0 e^{\zeta_-}\frac{\mathrm{sh}((b-1)\eta_-(x))}{\mathrm{sh}(b\eta_-(x))}\right)}$$

which in principle gives the probability function for N by Taylor expansion. Closed form expressions for the probability function can be given in some special cases, for example when $p_0 = p_+ = p_-$, see below. For numerical values of a, b, p_0, p_+ and p_- it is easy to get the Taylor expansion with for example the computer program Maple. Some examples are given in Figure 5.

For $p_0 = p_+ = p_-$

$$E\Big[x^N\Big] = \frac{e^{a\zeta}\mathrm{sh}(b\eta(x)) + e^{-b\zeta}\mathrm{sh}(a\eta(x))}{\mathrm{sh}((a+b)\eta(x))}$$

is a rational function, which gives the expansion

$$E\Big[x^N\Big] = \sum_{k=1}^{a+b-1}\frac{1}{x\sqrt{4pq}\cos\left(\frac{k\pi}{a+b}\right)-1}\frac{(-1)^k}{a+b}\sin\Big(\frac{k\pi}{a+b}\Big)\frac{1}{\cos\left(\frac{k\pi}{a+b}\right)}$$
$$\times\left(e^{a\zeta}\sin\left(\frac{kb\pi}{a+b}\right)+e^{-b\zeta}\sin\left(\frac{ka\pi}{a+b}\right)\right)$$

$$= \sum_{k=1}^{a+b-1}\sum_{j=0}^{\infty}\Big(x\sqrt{4pq}\cos\Big(\frac{k\pi}{a+b}\Big)\Big)^j\frac{(-1)^{k+1}}{a+b}\sin\Big(\frac{k\pi}{a+b}\Big)\frac{1}{\cos\left(\frac{k\pi}{a+b}\right)}$$
$$\times\left(e^{a\zeta}\sin\left(\frac{kb\pi}{a+b}\right)+e^{-b\zeta}\sin\left(\frac{ka\pi}{a+b}\right)\right),$$

and therefore

$$P(N=j) = \frac{\sqrt{4pq}^{\,j}}{a+b}\sum_{k=1}^{a+b-1}(-1)^{k+1}\sin\Big(\frac{k\pi}{a+b}\Big)\cos^{j-1}\left(\frac{k\pi}{a+b}\right)$$
$$\times\left(e^{a\zeta}\sin\left(\frac{kb\pi}{a+b}\right)+e^{-b\zeta}\sin\left(\frac{ka\pi}{a+b}\right)\right), \quad (15)$$

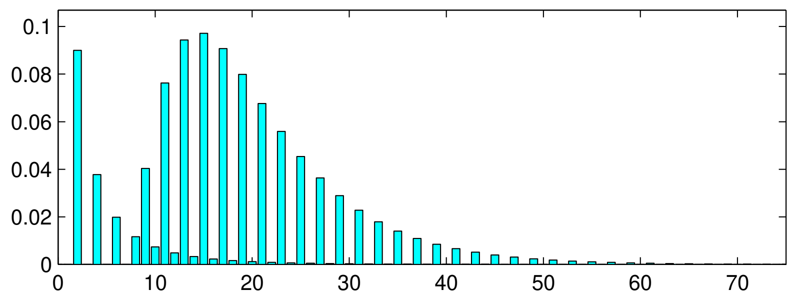


Figure 4: The probability function of N with $p = 0.7$, $a = 9$ and $b = 2$ (the tail is cut off at 75).

see Figure 4 for an example. Formula (15) can be found in the book by Feller [7] (in a different form) and has a long history going back to Lagrange.

To get the probability of absorption in a, $P(S_N = a)$, set $x = 1$ in (13):

$$E\left[1^{Z_{abs}} I_{S_N=a}\right] = E[I_{S_N=a}] = P(S_N = a) = \frac{p_0 e^{(a-1)\zeta_+} \frac{\mathrm{sh}(\zeta_+)}{\mathrm{sh}(a\zeta_+)}}{p_{abs}}$$

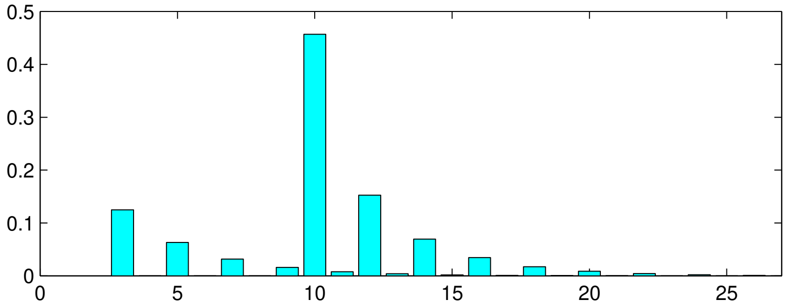
$$= \frac{p_0 p_+^{a-1} \frac{p_+ - q_+}{p_+^a - q_+^a}}{p_0 p_+^{a-1} \frac{p_+ - q_+}{p_+^a - q_+^a} + q_0 q_-^{b-1} \frac{p_- - q_-}{p_-^b - q_-^b}}. \quad (16)$$

In Figure 6 $P(S_N = a)$ is plotted for fixed values of a, b, p_0 and varying p_+, p_-.

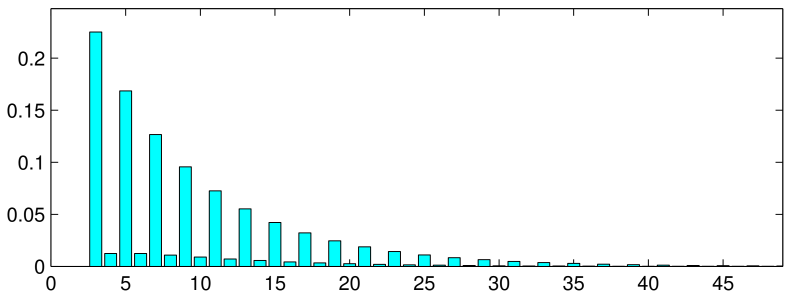
3 First passage time for a continuous time Markov chain with independent exponentially distributed holding times

In Section 2 the holding times were all constant and equal to 1. Now we let the holding times be independent exponentially distributed with an intensity depending on which state the Markov chain, $\{S_t\}_{t\geq 0}$, is in. In Figure 7 is a realization of $\{S_t\}_{t\geq 0}$ given.

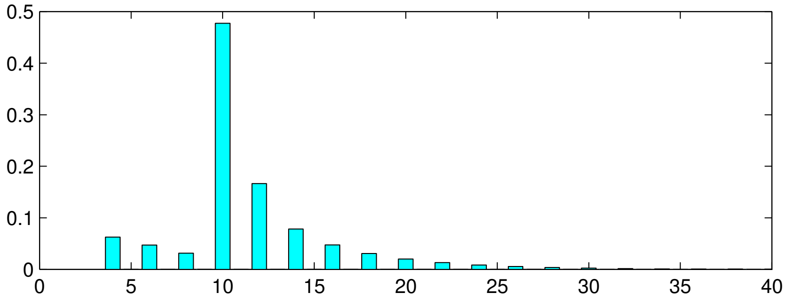
If S_t is in state 0 let the holding time have intensity $\lambda_0 + \mu_0 := \frac{1}{l_0}$ (or mean holding time l_0) with transition probabilities $p_0 = \lambda_0 l_0$ and $q_0 = \mu_0 l_0$. Similarly if the chain is in a positive or a negative state let the intensity be $\lambda_+ + \mu_+ := \frac{1}{l_+}$ respective $\lambda_- + \mu_- := \frac{1}{l_-}$ with transition probabilities $p_+ = \lambda_+ l_+$ and $q_+ = \mu_+ l_+$



(a) $a = 3, b = 10, p_0 = 0.5, p_+ = 0.5, p_- = 0.01$

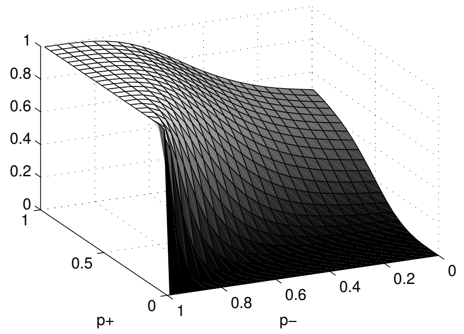


(b) $a = 3, b = 4, p_0 = 0.9, p_+ = 0.5, p_- = 0.5$

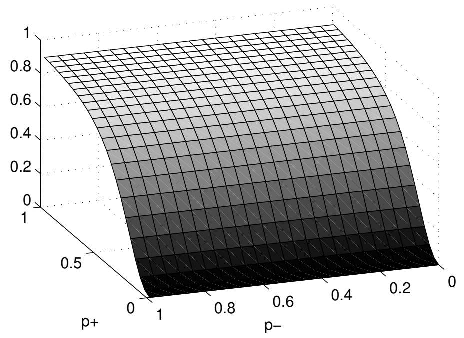


(c) $a = 4, b = 10, p_0 = 0.5, p_+ = 0.5, p_- = 0.01$

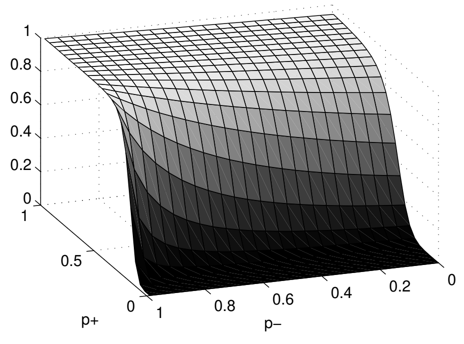
Figure 5: Example of the probability function of N. The tails are cut off.



(a) $a = 4, b = 4, p_0 = 0.5$



(b) $a = 3, b = 1, p_0 = 0.9$



(c) $a = 6, b = 2, p_0 = 0.95$

Figure 6: Plot of $P(S_N = a)$ for varying p_+ and p_-.

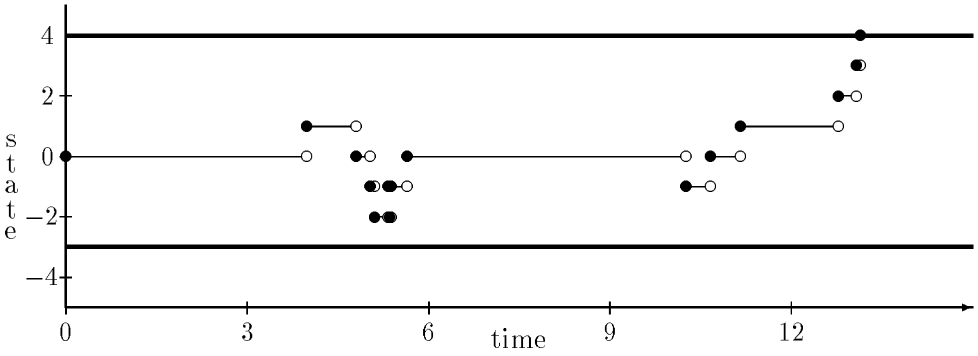


Figure 7: A realization of a continuous time Markov chain with state depending transition probabilities and borders $a = 4$ and $b = -3$. The mean holding time is 0.25 if S_t is in a negative state and 1 otherwise. Note that the jump chain is the same as in Figure 2, it is just the holding times that are changed.

respective $p_- = \lambda_- l_-$ and $q_- = \mu_- l_-$, see Figure 8. Let H_i be the holding time between the *i-1*th and the i^{th} jump, for $i = 1, 2, \dots$.

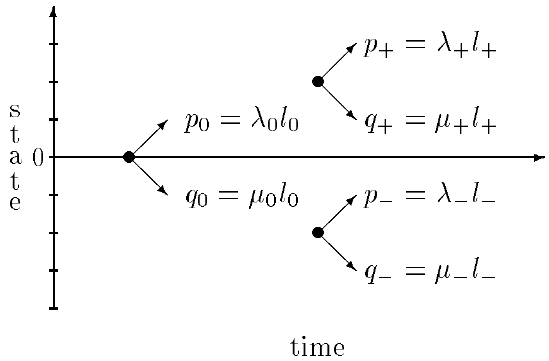


Figure 8: The transition probabilities for $\{S_t\}_{t\geq 0}$.

Let L be the time until hitting a border for the first time (L corresponds to N in Section 2)

$$L \stackrel{\text{def}}{=} \inf\{t : S_t = a \text{ or } S_t = -b\}.$$

Define L_i to be the length of the excursion between the *i-1*th and the i^{th} return to 0, $L_0 \equiv 0$ (L_i corresponds to Z_i in Section 2). Let L_{abs} be the length of the last excursion from 0 to either a or $-b$ (L_{abs} corresponds to Z_{abs} in Section 2). Let N_0

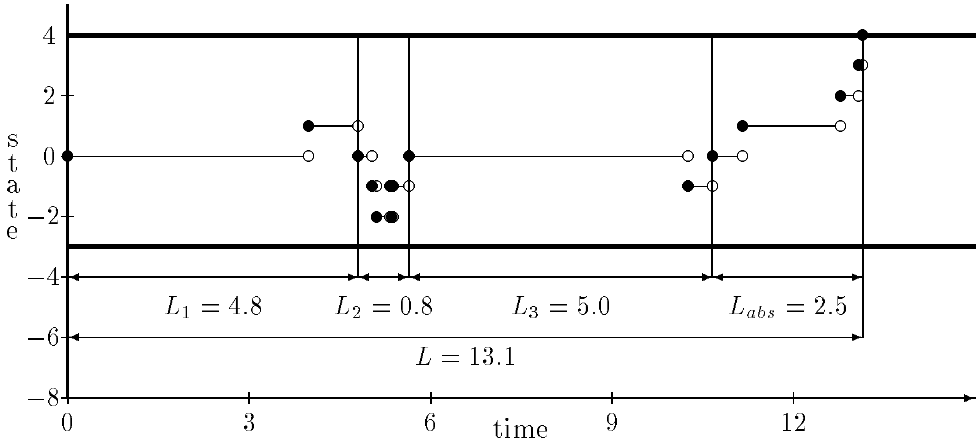


Figure 9: Realization of a continuous time Markov chain with state depending transition probabilities and borders $a = 4$ and $b = -3$. The parameters used are $\lambda_0 = 0.5$, $\mu_0 = 0.5$, $\lambda_+ = 0.5$, $\mu_+ = 0.5$, $\lambda_- = 2$ and $\mu_- = 2$.

be as before, (9). Now L can be expressed in terms of $\{L_i\}_{i\geq 0}$, L_{abs} and N_0 (see Figure 9):

$$L = \sum_{i=0}^{N_0} L_i + L_{abs}.$$

As in Section 2 we get by independence that the Laplace transform can be written

$$\begin{aligned} E\left[e^{-sL}\right] &= E\left[e^{-s\sum_{i=0}^{N_0} L_i}\right] E\left[e^{-sL_{abs}}\right] \\ &= E\left[E\left[e^{-sL_1}\right]^{N_0}\right]\left(E\left[e^{-sL_{abs}} I_{\{S_L=a\}}\right] + E\left[e^{-sL_{abs}} I_{\{S_L=b\}}\right]\right). \end{aligned} \quad (17)$$

By the same arguments as in Section 2

$$E\left[E\left[e^{-sL_1}\right]^{N_0}\right] = \frac{p_{abs}}{1 - q_{abs} E\left[e^{-sL_1}\right]}.$$

L_1 can be expressed in terms of H_k and Z_1 as

$$L_1 = \sum_{k=1}^{Z_1} H_k.$$

As the chain is conditioned on returning to 0 before visiting any of the borders a or $-b$ we get by conditioning on the the first jump that

$$
\begin{aligned}
E\left[e^{-sL_1}\right] &= E\left[e^{-s\sum_{k=1}^{Z_1} H_k}\right] = \frac{1}{1+sl_0} E\left[e^{-s\sum_{k=2}^{Z_1} H_k}\right] \\
&= \frac{1}{1+sl_0}\left(E\left[e^{-s\sum_{k=2}^{Z_1} H_k} | \tau_0^1 < \tau_0^{-1}\right] p_0 + E\left[e^{-s\sum_{k=2}^{Z_1} H_k} | \tau_0^1 > \tau_0^{-1}\right] q_0 \right) \\
&= \frac{1}{1+sl_0}\left(E_1\left[e^{-s\sum_{k=2}^{Z_1} H_k}\right] p_0 + E_{-1}\left[e^{-s\sum_{k=2}^{Z_1} H_k}\right] q_0 \right) \qquad (18)
\end{aligned}
$$

where the index on the expectations indicate where the process starts. In the first term of (18) all H_k are independent and have the same exponential distribution with mean l_+. We can use the same argument as in Section 2.1 to get

$$
E_1\left[e^{-s\sum_{k=2}^{Z_1} H_k}\right] = \frac{e^{-\zeta_+} \frac{\mathrm{sh}\left((a-1)\eta_+\left(\frac{1}{1+sl_+}\right)\right)}{\mathrm{sh}\left(a\eta_+\left(\frac{1}{1+sl_+}\right)\right)}}{q_{abs}},
$$

and

$$
E_{-1}\left[e^{-s\sum_{k=2}^{Z_1} H_k}\right] = \frac{e^{\zeta_-} \frac{\mathrm{sh}\left((b-1)\eta_-\left(\frac{1}{1+sl_-}\right)\right)}{\mathrm{sh}\left(b\eta_-\left(\frac{1}{1+sl_-}\right)\right)}}{q_{abs}}.
$$

Thus

$$
E\left[e^{-sL_1}\right] = \frac{1}{1+sl_0}\left(\frac{p_0 e^{-\zeta_+} \frac{\mathrm{sh}\left((a-1)\eta_+\left(\frac{1}{1+sl_+}\right)\right)}{\mathrm{sh}\left(a\eta_+\left(\frac{1}{1+sl_+}\right)\right)} + q_0 e^{\zeta_-} \frac{\mathrm{sh}\left((b-1)\eta_-\left(\frac{1}{1+sl_-}\right)\right)}{\mathrm{sh}\left(b\eta_-\left(\frac{1}{1+sl_-}\right)\right)}}{q_{abs}} \right).
$$

L_{abs} in the second part of (17) may also be expressed in terms of H_k:

$$
L_{abs} = \sum_{k=1}^{Z_{abs}} H_{k+\sum_{i=0}^{N_0} Z_i}.
$$

As before condition on the first step and use that H_k is exponential with mean l_+ and (13) in Section 2.2:

$$
E\left[e^{-sL_{abs}} I_{\{S_L=a\}}\right] = \frac{p_0 e^{(a-1)\zeta_+} \mathrm{sh}\left(\eta_+\left(\frac{1}{1+sl_+}\right)\right)}{(1+sl_0)p_{abs}\mathrm{sh}\left(a\eta_+\left(\frac{1}{1+sl_+}\right)\right)} \qquad (19)
$$

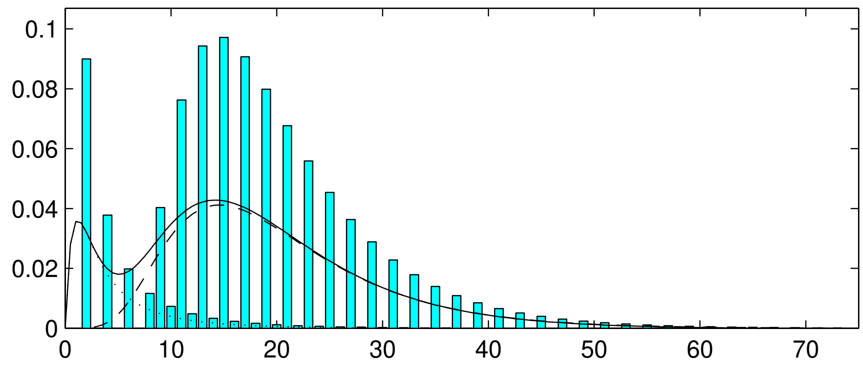


Figure 10: The probability function of N (bars) and the density of L (solid line) with $p = 0.7$, $a = 9$, $b = 2$ and $l_0 = l_+ = l_- = 1$ (the tails are cut off at 75). The areas under the dashed and the dotted lines corresponds to the probability to be absorbed in a respective $-b$.

and similarly

$$E\left[e^{-sL_{abs}}I_{\{S_L=b\}}\right] = \frac{q_0 e^{-(b-1)\zeta_-}\mathrm{sh}\left(\eta_-\left(\frac{1}{1+sl_-}\right)\right)}{(1+sl_0)p_{abs}\mathrm{sh}\left(b\eta_-\left(\frac{1}{1+sl_-}\right)\right)}.$$

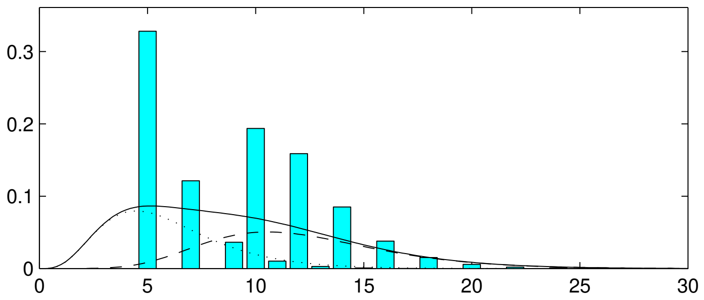
Combining the formulas above gives the Laplace transform

$$E\left[e^{-sL}\right] = \frac{\frac{p_0}{1+sl_0}e^{(a-1)\zeta_+}\frac{\mathrm{sh}(\chi_+(sl_+))}{\mathrm{sh}(a\chi_+(sl_+))} + \frac{q_0}{1+sl_0}e^{-(b-1)\zeta_-}\frac{\mathrm{sh}(\chi_-(sl_-))}{\mathrm{sh}(b\chi_-(sl_-))}}{1 - \frac{1}{1+sl_0}\left(p_0 e^{-\zeta_+}\frac{\mathrm{sh}((a-1)\chi_+(sl_+))}{\mathrm{sh}(a\chi_+(sl_+))} + q_0 e^{\zeta_-}\frac{\mathrm{sh}((b-1)\chi_-(sl_-))}{\mathrm{sh}(b\chi_-(sl_-))}\right)} \tag{20}$$

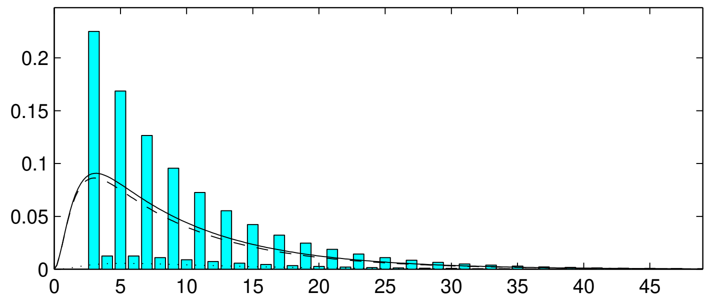
where $\zeta_\pm = \ln\sqrt{\frac{p_\pm}{q_\pm}}$ and $\chi_\pm(sl_\pm) = \eta_\pm\left(\frac{1}{1+sl_\pm}\right) = \mathrm{arccosh}\left(\frac{1+sl_\pm}{\sqrt{4p_\pm q_\pm}}\right)$.

Analytical inversion of this complicated Laplace transform to get the distribution of L does not seem feasible. If the holding times are identically distributed it is possible to use the explicit formula for $P(N = n)$ in Section 2.3 to derive the density of L without using the Laplace transform, see Figure 10. But if the holding times have different distributions this seems impossible. However there are several numerical methods to invert Laplace transforms. One by Abate and Whitt [2] is described in the Appendix. In Figure 11 some examples are given of the density of L where the inversion of the Laplace transform is done by this method.

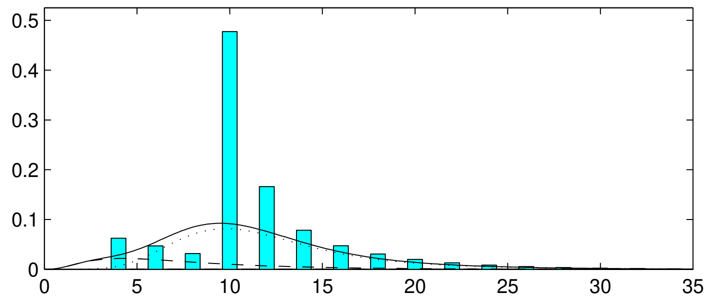
Even though the holding times are different from the random walk case the probability of absorption at a is the same.



(a) $a = 5, b = 10, p_0 = 0.5, p_+ = 0.9, p_- = 0.1$



(b) $a = 3, b = 4, p_0 = 0.9, p_+ = 0.5, p_- = 0.5$



(c) $a = 4, b = 10, p_0 = 0.5, p_+ = 0.5, p_- = 0.01$

Figure 11: The probability function of N (bars) and the density of L (solid line). The mean holding times are 1. The areas under the dashed and the dotted lines corresponds to the probability to be absorbed in a respective $-b$. The tails are cut off.

4 The limit of a scaled continuous Markov chain with sign depending stochastic holding times

If the jumps and the time are scaled appropriately, the continuous Markov chain tends to a limit. To obtain an appropriate scaling consider the process above the x-axis. Let $\{X_t\}_{t\geq 0}$ and $\{Y_t\}_{t\geq 0}$ be Poisson processes with intensity λ_+ respective μ_+. Then $\{X_t - Y_t\}_{t\geq 0}$ is a birth and death process with intensity λ_+ to jump one step up and intensity μ_+ to jump one step down. This can be considered as the process above the x-axis. The scaled process $\{c(X_t - Y_t); t \geq 0\}$ has

$$\begin{cases} \mathrm{E}[c(X_t - Y_t)] = c(\lambda_+ - \mu_+)t = \frac{c(p_+ - q_+)t}{l_+} := \nu_+ t \\ \mathrm{Var}(c(X_t - Y_t)) = c^2(\lambda_+ + \mu_+)t = \frac{c^2 t}{l_+} := \sigma_+^2 t \end{cases},$$

i.e. with drift ν_+ and variance σ_+^2. The process below the x-axis can be considered similarly and we get

$$\begin{cases} p_\pm &= \frac{1}{2}\left(1 + \frac{c\nu_\pm}{\sigma_\pm^2}\right) \\ l_\pm &= \frac{c^2}{\sigma_\pm^2}. \end{cases}$$

Theorem 4.1 *Set* $\mathrm{sh}(\cdot) = \sinh(\cdot)$, $\mathrm{ch}(\cdot) = \cosh(\cdot)$, $\zeta_\pm = \ln\sqrt{\frac{p_\pm}{q_\pm}}$ *and* $\chi_\pm(sl_\pm) = \operatorname{arccosh}\left(\frac{1+sl_\pm}{\sqrt{4p_\pm q_\pm}}\right)$. *Let* $c \to 0$, $l_\pm \to 0$, $l_0 \to 0$, $p_\pm \to \frac{1}{2}$, $a, b \to \infty$ *with fixed* $\nu_\pm$, $\sigma_\pm^2$, $\frac{c^2}{l_0} > 0$, $r_a = ac > 0$, $r_b = bc > 0$. *Then*

$$\begin{aligned} E\left[e^{-sL}\right] &= \frac{\frac{p_0}{1+sl_0} e^{(a-1)\zeta_+} \frac{\mathrm{sh}(\chi_+(sl_+))}{\mathrm{sh}(a\chi_+(sl_+))} + \frac{q_0}{1+sl_0} e^{-(b-1)\zeta_-} \frac{\mathrm{sh}(\chi_-(sl_-))}{\mathrm{sh}(b\chi_-(sl_-))}}{1 - \frac{1}{1+sl_0}\left(p_0 e^{-\zeta_+} \frac{\mathrm{sh}((a-1)\chi_+(sl_+))}{\mathrm{sh}(a\chi_+(sl_+))} + q_0 e^{\zeta_-} \frac{\mathrm{sh}((b-1)\chi_-(sl_-))}{\mathrm{sh}(b\chi_-(sl_-))}\right)} \\ &\to \frac{p_0 \frac{\alpha_a(s)\,\mathrm{sh}(\alpha_b(s))}{r_a} e^{\varepsilon_+} + q_0 \frac{\alpha_b(s)\,\mathrm{sh}(\alpha_a(s))}{r_b} e^{-\varepsilon_-}}{p_0\,\mathrm{sh}(\alpha_b(s)) \frac{\alpha_a(s)\,\mathrm{ch}(\alpha_a(s)) + \varepsilon_+\,\mathrm{sh}(\alpha_a(s))}{r_a} + q_0\,\mathrm{sh}(\alpha_a(s)) \frac{\alpha_b(s)\,\mathrm{ch}(\alpha_b(s)) - \varepsilon_-\,\mathrm{sh}(\alpha_b(s))}{r_b}}, \end{aligned}$$

where

$$\alpha_a(s) := \frac{r_a}{\sigma_+}\sqrt{2s + \frac{\nu_+^2}{\sigma_+^2}} \qquad \alpha_b(s) := \frac{r_b}{\sigma_-}\sqrt{2s + \frac{\nu_-^2}{\sigma_-^2}}$$

and

$$\varepsilon_+ = r_a \frac{\nu_+}{\sigma_+^2} \qquad \varepsilon_- = r_b \frac{\nu_-}{\sigma_-^2}.$$

To prove Theorem 4.1 we need some limits.

Lemma 4.1 *For fixed $r_d, s > 0$ and $r_d = dc$, as $c \to 0$,*

$$\chi(sl) = \frac{c}{\sigma}\sqrt{2s + \frac{s^2c^2}{\sigma^2} + \frac{\nu^2}{\sigma^2}} + O(c^2) \to 0, \tag{i}$$

$$\zeta = \frac{c\nu}{\sigma^2} + O(c^2) \to 0, \tag{ii}$$

$$d\chi(sl) \to \frac{r_d}{\sigma}\sqrt{2s + \frac{\nu^2}{\sigma^2}} = \alpha_d(s), \tag{iii}$$

$$d\zeta \to \frac{r_d\nu}{\sigma^2} = \varepsilon. \tag{iv}$$

Proof. The first limit, (i), is proved in [3] equation number (13). The second, (ii), is proved by Taylor expansion:

$$\zeta = \ln\sqrt{\frac{p}{q}} = \frac{1}{2}\Big[\ln\Big(\frac{1}{2}\Big(1 + \frac{c\nu}{\sigma^2}\Big)\Big) - \ln\Big(\frac{1}{2}\Big(1 - \frac{c\nu}{\sigma^2}\Big)\Big)\Big]$$
$$= \frac{1}{2}\Big[\frac{c\nu}{\sigma^2} + O(c^2) - \Big(-\frac{c\nu}{\sigma^2} + O(c^2)\Big)\Big] = \frac{c\nu}{\sigma^2} + O(c^2).$$

Using (i) and that $r_d = cd$, then as $c \to 0$

$$d\chi(sl) = \frac{r_d}{\sigma}\sqrt{2s + \frac{s^2c^2}{\sigma^2} + \frac{\nu^2}{\sigma^2}} + O(c) \to \frac{r_d}{\sigma}\sqrt{2s + \frac{\nu^2}{\sigma^2}}$$

and (iii) is proved. Using (ii) and that $r_d = cd$, (iv) is proved:

$$d\zeta = \frac{r_d\nu}{\sigma^2} + O(c) \to \frac{r_d\nu}{\sigma^2}$$

as $c \to 0$. □

Proof of Theorem 4.1. The Laplace transform of L can be arranged as

$$\frac{\frac{p_0}{1+sl_0}e^{(a-1)\zeta_+}\frac{\mathrm{sh}(\chi_+(sl_+))}{\mathrm{sh}(a\chi_+(sl_+))} + \frac{q_0}{1+sl_0}e^{-(b-1)\zeta_-}\frac{\mathrm{sh}(\chi_-(sl_-))}{\mathrm{sh}(b\chi_-(sl_-))}}{1 - \frac{1}{1+sl_0}\Big(p_0e^{-\zeta_+}\frac{\mathrm{sh}((a-1)\chi_+(sl_+))}{\mathrm{sh}(a\chi_+(sl_+))} + q_0e^{\zeta_-}\frac{\mathrm{sh}((b-1)\chi_-(sl_-))}{\mathrm{sh}(b\chi_-(sl_-))}\Big)}$$
$$= \Big\{\frac{p_0}{1+sl_0}e^{(a-1)\zeta_+}\mathrm{sh}(\chi_+(sl_+))\mathrm{sh}(b\chi_-(sl_-))$$
$$+ \frac{q_0}{1+sl_0}e^{-(b-1)\zeta_-}\mathrm{sh}(\chi_-(sl_-))\mathrm{sh}(a\chi_+(sl_+))\Big\}$$
$$\Big/\Big\{p_0\mathrm{sh}(b\chi_-(sl_-))\Big(\mathrm{sh}(a\chi_+(sl_+)) - \frac{e^{-\zeta_+}}{1+sl_0}\mathrm{sh}((a-1)\chi_+(sl_+))\Big)$$
$$+ q_0\mathrm{sh}(a\chi_+(sl_+))\Big(\mathrm{sh}(b\chi_-(sl_-)) - \frac{e^{\zeta_-}}{1+sl_0}\mathrm{sh}((b-1)\chi_-(sl_-))\Big)\Big\} \tag{21}$$

Both the numerator and the denominator in (21) tends to 0 when $c \to 0$, therefore scale them by $\frac{1}{c}$. Assume that $l_0 \to 0$ of the same order as l_+ and l_-, i.e. $l_0 K = c^2$ where $K > 0$ is some constant. Using (ii), (iii), (iv) and equation (16) in paper [3] the numerator divided by c tends to

$$p_0 \frac{\alpha_a(s)\mathrm{sh}(\alpha_b(s))}{r_a} e^{\varepsilon_+} + q_0 \frac{\alpha_b(s)\mathrm{sh}(\alpha_a(s))}{r_b} e^{-\varepsilon_-} \text{ as } c \to 0. \tag{22}$$

Start with the expression inside the parents in the first term in the denominator in (21) divided by c and use the definition of $\sinh(\cdot)$ then

$$\begin{aligned} &\frac{1}{c}\Big(\mathrm{sh}(a\chi_+(sl_+)) - \frac{e^{-\zeta_+}}{1+sl_0}\mathrm{sh}((a-1)\chi_+(sl_+))\Big) \\ &\quad = \frac{1}{2c}\left(e^{a\chi_+(sl_+)} - e^{-a\chi_+(sl_+)} - \frac{e^{-\zeta_+}}{1+sl_0}\Big(e^{(a-1)\chi_+(sl_+)} - e^{-(a-1)\chi_+(sl_+)}\Big)\right) \\ &= \frac{1}{2}\left(e^{(a-1)\chi_+(sl_+)}\frac{1}{c}\Big(e^{\chi_+(sl_+)} - \frac{e^{-\zeta_+}}{1+sl_0}\Big) + e^{-(a-1)\chi_+(sl_+)}\frac{1}{c}\Big(\frac{e^{-\zeta_+}}{1+sl_0} - e^{-\chi_+(sl_+)}\Big)\right). \end{aligned} \tag{23}$$

Use (i), (ii), (iii) and (iv) in Lemma 4.1 then the expression inside the parents in the first term in (23) divided by c becomes

$$\begin{aligned} \frac{1}{c}\left(e^{\chi_+(sl_+)} - \frac{e^{-\zeta_+}}{1+sl_0}\right) &= \frac{1}{c}\Big(1 + \chi_+(sl_+) + O(c^2) - \frac{1}{1+sl_0}(1 - \zeta_+ + O(c^2))\Big) \\ &= \frac{a\chi_+(sl_+)}{r_a} + \frac{a\zeta_+}{r_a(1+sl_0)} + \frac{1}{c}\frac{s\frac{c^2}{K}}{1+s\frac{c^2}{K}} + O(c) \\ &\to \frac{\alpha_a(s)}{r_a} + \frac{\varepsilon_+}{r_a} + 0 + 0 = \frac{\alpha_a(s) + \varepsilon_+}{r_a} \quad \text{as} \quad c \to 0. \end{aligned} \tag{24}$$

Similarly the expression inside the parents in the second term in (23) divided by c becomes

$$\begin{aligned} \frac{1}{c}\left(\frac{e^{-\zeta_+}}{1+sl_0} - e^{-\chi_+(sl_+)}\right) &= -\frac{1}{c}\left(e^{-\chi_+(sl_+)} - \frac{e^{-\zeta_+}}{1+sl_0}\right) \\ &= -\Big(-\frac{a\chi_+(sl_+)}{r_a} + \frac{a\zeta_+}{r_a(1+sl_0)} + \frac{1}{c}\frac{s\frac{c^2}{K}}{1+s\frac{c^2}{K}} + O(c)\Big) \\ &\to -\Big(-\frac{\alpha_a(s)}{r_a} + \frac{\varepsilon_+}{r_a} + 0 + 0\Big) = \frac{\alpha_a(s) - \varepsilon_+}{r_a} \quad \text{as} \quad c \to 0. \end{aligned} \tag{25}$$

Use (i) and (iii) in Lemma 4.1 and (24) and (25), then (23) tends to

$$\frac{1}{2}\left(e^{\alpha_a(s)}\Big(\frac{\alpha_a(s)+\varepsilon_+}{r_a}\Big)+e^{-\alpha_a(s)}\Big(\frac{\alpha_a(s)-\varepsilon_+}{r_a}\Big)\right)$$
$$=\frac{\alpha_a(s)\mathrm{ch}(\alpha_a(s))+\varepsilon_+\mathrm{sh}(\alpha_a(s))}{r_a} \quad (26)$$

as $c \to 0$. Similarly the expression inside the parents in the second term in the denominator in (21) divided by c becomes

$$\frac{1}{2c}\left(e^{(b-1)\chi_-(sl_-)}\Big(e^{\chi_-(sl_-)}-\frac{e^{\zeta_-}}{1+sl_0}\Big)+e^{-(b-1)\chi_-(sl_-)}\Big(\frac{e^{\zeta_-}}{1+sl_0}-e^{-\chi_-(sl_-)}\Big)\right)$$
$$\to\frac{1}{2}\left(e^{\alpha_b(s)}\Big(\frac{\alpha_b(s)-\varepsilon_-}{r_b}\Big)+e^{-\alpha_b(s)}\Big(\frac{\alpha_b(s)+\varepsilon_-}{r_b}\Big)\right)$$
$$=\frac{\alpha_b(s)\mathrm{ch}(\alpha_b(s))-\varepsilon_-\mathrm{sh}(\alpha_b(s))}{r_b} \quad \text{as} \quad c\to 0. \quad (27)$$

Hence the denominator in (21) divided by c tends to

$$p_0\mathrm{sh}(\alpha_b(s))\frac{\alpha_a(s)\mathrm{ch}(\alpha_a(s))+\varepsilon_+\mathrm{sh}(\alpha_a(s))}{r_a}$$
$$+\,q_0\mathrm{sh}(\alpha_a(s))\frac{\alpha_b(s)\mathrm{ch}(\alpha_b(s))-\varepsilon_-\mathrm{sh}(\alpha_b(s))}{r_b} \quad (28)$$

when $c \to 0$. Finally the Laplace transform of L, (21), tends to (22) divided by (28):

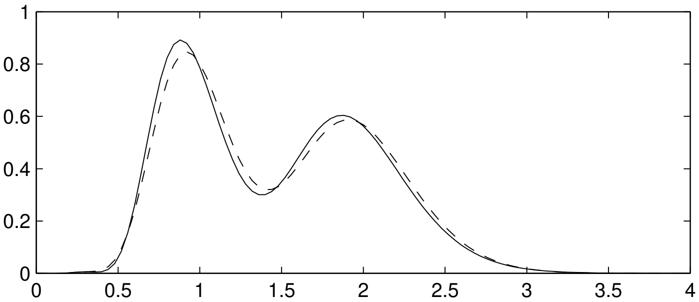
$$\frac{p_0\frac{\alpha_a(s)\mathrm{sh}(\alpha_b(s))}{r_a}e^{\varepsilon_+}+q_0\frac{\alpha_b(s)\mathrm{sh}(\alpha_a(s))}{r_b}e^{-\varepsilon_-}}{p_0\mathrm{sh}(\alpha_b(s))\frac{\alpha_a(s)\mathrm{ch}(\alpha_a(s))+\varepsilon_+\mathrm{sh}(\alpha_a(s))}{r_a}+q_0\mathrm{sh}(\alpha_a(s))\frac{\alpha_b(s)\mathrm{ch}(\alpha_b(s))-\varepsilon_-\mathrm{sh}(\alpha_b(s))}{r_b}}$$

as $c \to 0$. □

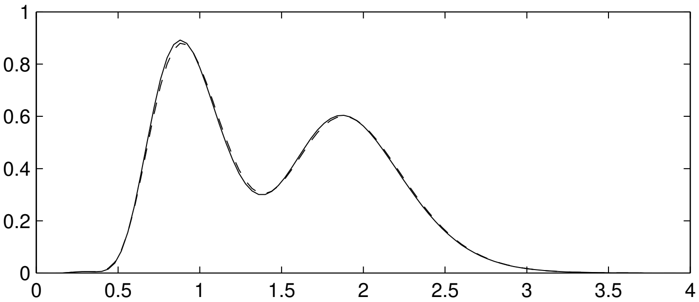
An example of the convergence of the absorption time, L, to the limit distribution is given in Figure 12 and Figure 13 and the parameters used is given in Table 1 and Figure 13 and Table 2. The inversions of the Laplace transforms are done by the method described in the Appendix.

The probability that the limit process is absorbed in a can be derived from (16):

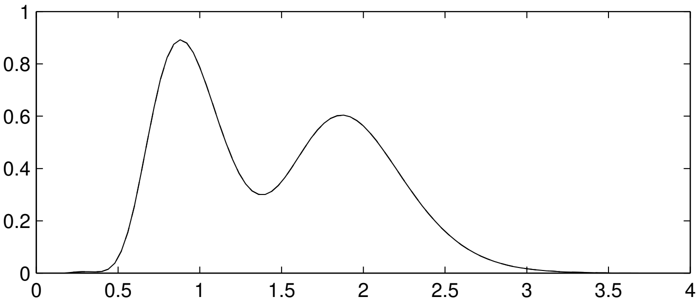
$$P(S_L=a)=\frac{p_0e^{(a-1)\zeta_+}\frac{\mathrm{sh}(\zeta_+)}{\mathrm{sh}(a\zeta_+)}}{p_0e^{(a-1)\zeta_+}\frac{\mathrm{sh}(\zeta_+)}{\mathrm{sh}(a\zeta_+)}+q_0e^{-(b-1)\zeta_-}\frac{\mathrm{sh}(\zeta_-)}{\mathrm{sh}(b\zeta_-)}}.$$



(a) $c = 0.1$

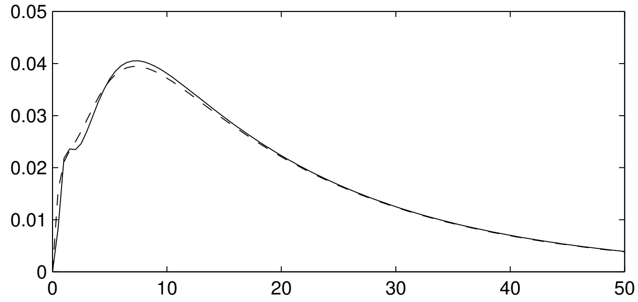


(b) $c = 0.05$

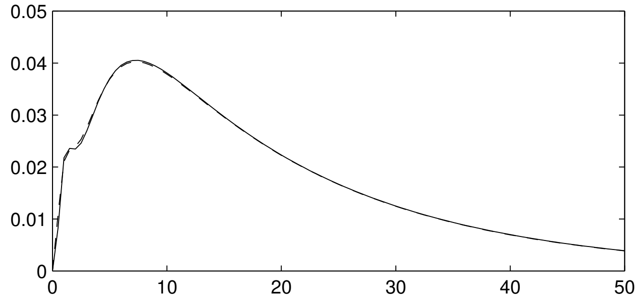


(c) $c = 0.01$

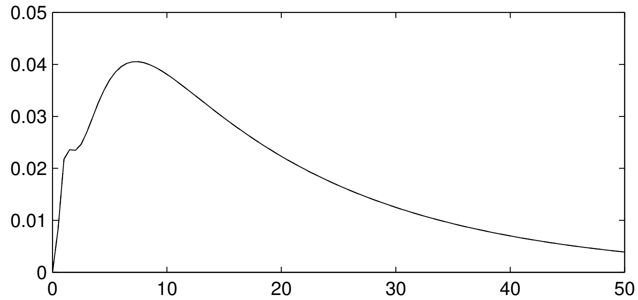
Figure 12: Example of the convergence of the absorption time L. The solid line is the limit density and the dashed line is the density for the absorption time of a scaled continuous time Markov chain. The parameters are given in Table 1.



(a) $c = 1$



(b) $c = 0.5$



(c) $c = 0.1$

Figure 13: Example of the convergence of the absorption time L. The solid line is the limit density and the dashed line is the density for the absorption time of a scaled continuous time Markov chain. The parameters are given in Table 2.

c	l_+	l_-	p_+	p_-	Figure
0.10	0.04	0.04	0.90	0.10	12(a)
0.05	0.01	0.01	0.70	0.30	12(b)
0.01	0.0004	0.0004	0.54	0.46	12(c)

Table 1: Parameters: $p_0 = 0.5$, $r_a = 2$, $r_b = 4$, $\nu_+ = 2$, $\nu_- = -2$, $\sigma_+^2 = \sigma_-^2 = 0.25$.

c	l_+	l_-	p_+	p_-	Figure
1	1	1	0.5	0.5	13(a)
0.5	0.25	0.25	0.5	0.5	13(b)
0.1	0.01	0.01	0.5	0.5	13(c)

Table 2: Parameters: $p_0 = 0.9$, $r_a = 5$, $r_b = 2$, $\nu_+ = \nu_- = 0$, $\sigma_+^2 = \sigma_-^2 = 1$.

We know from Lemma 4.1, (iv), that $a\zeta_+ \to \varepsilon_+$ and $b\zeta_- \to \varepsilon_-$ as $c \to 0$. Because both the denominator and the numerator tend to 0, we divide both by c then

$$\begin{aligned}\frac{1}{c}\mathrm{sh}(\zeta) = \frac{1}{2c}(e^{\zeta} - e^{-\zeta}) &= \frac{e^{-\zeta}}{2c}(e^{2\zeta} - 1) \\ &= \frac{e^{-\zeta}}{2c}(1 + 2\zeta + O(c^2) - 1) = e^{-\zeta}(\frac{\nu}{\sigma^2} + O(c)) \to \frac{\nu}{\sigma^2} \quad \text{as} \quad c \to 0.\end{aligned}$$

This leads to

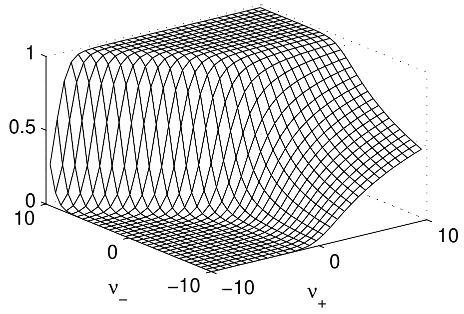
$$P(S_L = a) \to \frac{\frac{p_0\nu_+ e^{\varepsilon_+}}{\sigma_+^2 \mathrm{sh}(\varepsilon_+)}}{\frac{p_0\nu_+ e^{\varepsilon_+}}{\sigma_+^2 \mathrm{sh}(\varepsilon_+)} + \frac{q_0\nu_- e^{-\varepsilon_-}}{\sigma_-^2 \mathrm{sh}(\varepsilon_-)}}$$

when $c \to 0$ and the scaling in Theorem 4.1 is used. Three examples of the probability of absorption in r_a are given in Figure 14.

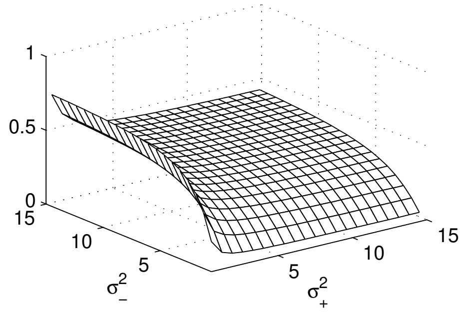
5 Extensions of first passage time

5.1 Time spent on the x-axis

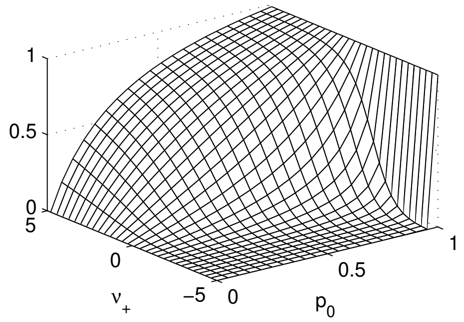
The time the continuous time Markov chain spends on the x-axis, in state 0, can be expressed as $\sum_{i=1}^{N_0} H_i + H_{abs}$ where all H_i and H_{abs} are independent exponentially distributed random variables with mean l_0. The Laplace transform for the time



(a) $r_a = r_b = \sigma_+^2 = \sigma_-^2 = 1$, $p_0 = 0.5$



(b) $r_a = r_b = 1 = \nu_+ = 1$, $\nu_- = -4$, $p_0 = 0.5$



(c) $r_a = r_b = \sigma_+^2 = \sigma_-^2 = 1$, $\nu_- = -1$

Figure 14: The probability to be absorbed in r_a.

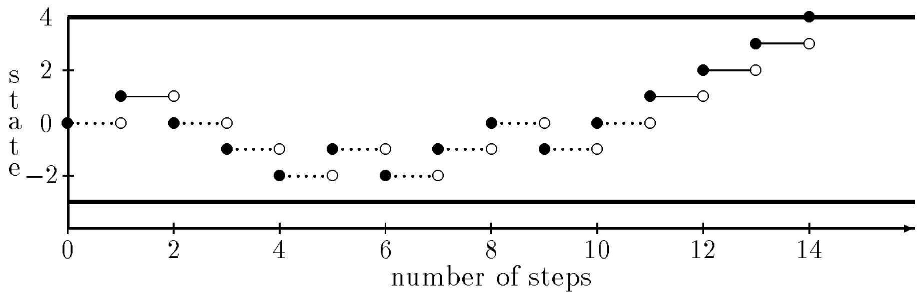


Figure 15: The solid lines represents the time the Markov chain spends above the x-axis.

spent on the x-axis is

$$E\left[e^{-s\left(\sum_{i=1}^{N_0} H_i + H_{abs}\right)}\right] = \frac{1}{1+sl_0} E\left[e^{-s\sum_{i=1}^{N_0} H_i}\right] = \frac{1}{1+sl_0} E\left[E\left[e^{-sH_1}\right]^{N_0}\right]$$

$$= \frac{1}{(1+sl_0)} \frac{p_{abs}}{\left(1 - q_{abs}E[e^{-sH_1}]\right)} = \frac{p_{abs}}{(1+sl_0)\left(1 - \frac{q_{abs}}{1+sl_0}\right)} = \frac{p_{abs}}{p_{abs}+sl_0} = \frac{1}{1+\frac{sl_0}{p_{abs}}}$$

Which is the Laplace transform for an exponentially distributed random variable with mean $\frac{l_0}{p_{abs}}$.

5.2 Time spent above the x-axis

Let U_N denote the time the discrete Markov chain spends above the x-axis before absorption, see Figure 15, then

$$E\left[x^{U_N}\right] = E\left[x^{\sum_{i=0}^{N_0}(Z_i-1)1_{A_i} + (Z_{abs}-1)1_{\{S_N=a\}}}\right] \tag{29}$$

where

$$1_{A_i} = \begin{cases} 1 & \text{if the i}^{\text{th}} \text{ excursion path is above 0} \\ 0 & \text{otherwise.} \end{cases}$$

By independence (29) can be written as

$$E\left[x^{\sum_{i=0}^{N_0}(Z_i-1)1_{A_i}}\right] E\left[x^{(Z_{abs}-1)1_{\{S_N=a\}}}\right] = E\left[E\left[x^{(Z_1-1)1_{A_1}}\right]^{N_0}\right] E\left[x^{(Z_{abs}-1)1_{\{S_N=a\}}}\right].$$

From (11) and (14)

$$E\left[x^{(Z_1-1)1_{A_1}}\right] = E\left[x^{(Z_1-1)1_{\{S_1=1\}}}\right]$$
$$= \frac{p_0}{q_{abs}} e^{-\zeta_+} \frac{\mathrm{sh}((a-1)\eta_+(x))}{\mathrm{sh}(a\eta_+(x))} + \frac{q_0}{q_{abs}} e^{\zeta_-} \frac{\mathrm{sh}((b-1)\zeta_-)}{\mathrm{sh}(b\zeta_-)}$$

and

$$E\left[x^{(Z_{abs}-1)1_{\{S_N=a\}}}\right] = \frac{p_0}{p_{abs}} e^{(a-1)\zeta_+} \frac{\mathrm{sh}(\eta_+(x))}{\mathrm{sh}(a\eta_+(x))} + \frac{q_0}{p_{abs}} e^{-(b-1)\zeta_-} \frac{\mathrm{sh}(\zeta_-)}{\mathrm{sh}(b\zeta_-)}$$

and N_0 is geometrically distributed, see (9), hence

$$\begin{aligned} E\left[x^{U_N}\right] &= \sum_{n=0}^{\infty}\Big(p_0 e^{-\zeta_+} \frac{\mathrm{sh}((a-1)\eta_+(x))}{\mathrm{sh}(a\eta_+(x))} + q_0 e^{\zeta_-} \frac{\mathrm{sh}((b-1)\zeta_-)}{\mathrm{sh}(b\zeta_-)}\Big)^n \\ &\quad \times p_{abs}\Big(\frac{p_0}{p_{abs}} e^{(a-1)\zeta_+} \frac{\mathrm{sh}(\eta_+(x))}{\mathrm{sh}(a\eta_+(x))} + \frac{q_0}{p_{abs}} e^{-(b-1)\zeta_-} \frac{\mathrm{sh}(\zeta_-)}{\mathrm{sh}(b\zeta_-)}\Big) \\ &= \frac{p_0 e^{(a-1)\zeta_+} \frac{\mathrm{sh}(\eta_+(x))}{\mathrm{sh}(a\eta_+(x))} + q_0 e^{-(b-1)\zeta_-} \frac{\mathrm{sh}(\zeta_-)}{\mathrm{sh}(b\zeta_-)}}{1 - \Big(p_0 e^{-\zeta_+} \frac{\mathrm{sh}((a-1)\eta_+(x))}{\mathrm{sh}(a\eta_+(x))} + q_0 e^{\zeta_-} \frac{\mathrm{sh}((b-1)\zeta_-)}{\mathrm{sh}(b\zeta_-)}\Big)}. \end{aligned} \tag{30}$$

Let U denote the time the continuous time Markov chain spends above the x-axis before absorption then

$$E\left[e^{-sU}\right] = E\left[e^{-sU} 1_{\{U=0\}}\right] + E\left[e^{-sU} 1_{\{U>0\}}\right].$$

The distribution consists of two parts, one continuous part and one point mass (when $U = 0$). As the holding times above 0 are independent exponentially distributed with mean l_+ then

$$U = \sum_{i=0}^{U_N} H_i$$

and

$$\begin{aligned} E\left[e^{-sU}\right] &= E\left[e^{-s\sum_{i=0}^{U_N} H_i}\right] = E\left[E\left[e^{-sH_i}\right]^{U_N}\right] = E\left[E\left[\frac{1}{1+sl_+}\right]^{U_N}\right] \\ &= \frac{p_0 e^{(a-1)\zeta_+} \frac{\mathrm{sh}(\chi_+(s))}{\mathrm{sh}(a\chi_+(s))} + q_0 e^{-(b-1)\zeta_-} \frac{\mathrm{sh}(\zeta_-)}{\mathrm{sh}(b\zeta_-)}}{1 - \Big(p_0 e^{-\zeta_+} \frac{\mathrm{sh}((a-1)\chi_+(s))}{\mathrm{sh}(a\chi_+(s))} + q_0 e^{\zeta_-} \frac{\mathrm{sh}((b-1)\zeta_-)}{\mathrm{sh}(b\zeta_-)}\Big)}. \end{aligned}$$

where the last equality comes from (30). For the point mass in 0

$$E\left[e^{-sU}1_{\{U=0\}}\right] = P(U=0) = P(U_N=0)$$
$$= P(S_i \leq 0,\ \forall i \leq N) = \frac{q_0 e^{-(b-1)\zeta_-}\frac{\mathrm{sh}(\zeta_-)}{\mathrm{sh}(b\zeta_-)}}{1 - q_0 e^{\zeta_-}\frac{\mathrm{sh}((b-1)\zeta_-)}{\mathrm{sh}(b\zeta_-)}}$$

hence

$$E\left[e^{-sU}1_{\{U>0\}}\right] = \frac{p_0 e^{(a-1)\zeta_+}\frac{\mathrm{sh}(\chi_+(s))}{\mathrm{sh}(a\chi_+(s))} + q_0 e^{-(b-1)\zeta_-}\frac{\mathrm{sh}(\zeta_-)}{\mathrm{sh}(b\zeta_-)}}{1 - \left(p_0 e^{-\zeta_+}\frac{\mathrm{sh}((a-1)\chi_+(s))}{\mathrm{sh}(a\chi_+(s))} + q_0 e^{\zeta_-}\frac{\mathrm{sh}((b-1)\zeta_-)}{\mathrm{sh}(b\zeta_-)}\right)}$$
$$- \frac{q_0 e^{-(b-1)\zeta_-}\frac{\mathrm{sh}(\zeta_-)}{\mathrm{sh}(b\zeta_-)}}{1 - q_0 e^{\zeta_-}\frac{\mathrm{sh}((b-1)\zeta_-)}{\mathrm{sh}(b\zeta_-)}}.$$

The continuous part is possible to invert by the method described in appendix. In Figure 16 are two examples given of the density of the time spent above the x-axis.

5.3 The time from the last visit to 0 until absorption

The continuous time Markov chain will behave in the same way until the point when it leaves the x-axis for the last time, see Figure 17, independent of whether the end point is a or $-b$. This Section investigates conditioned on the end point the time from when the process leaves the x-axis for the last time until absorption. We have that

$$E\left[e^{-s(L_{abs}-H)}I_{S_L=a}\right] = E\left[e^{-s(L_{abs}-H)}I_{S_L=a}|S_L=a\right]P(S_L=a)$$

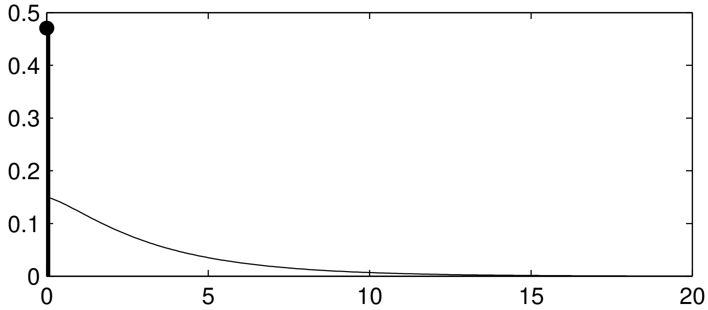
where H is the time spent on the x-axis the last time and it follows that

$$E\left[e^{-s(L_{abs}-H)}|S_L=a\right] = \frac{E\left[e^{-s(L_{abs}-H)}I_{S_L=a}\right]}{P(S_L=a)} = \frac{\mathrm{sh}(\chi_+(sl_+))\mathrm{sh}(a\zeta_+)}{\mathrm{sh}(a\chi_+(sl_+))\mathrm{sh}(\zeta_+)} \tag{31}$$

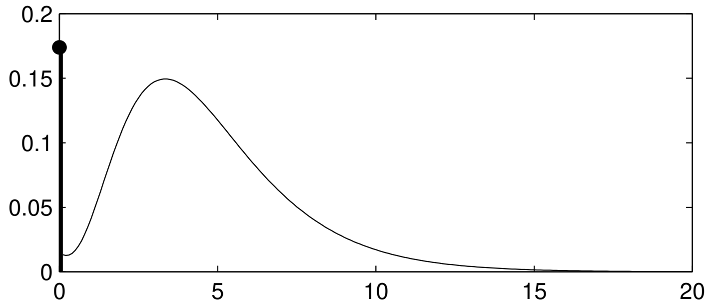
from (19) and (16). With the scaling in Section 4 as c tends to 0 (31) tends to

$$\frac{\alpha_a(s)\mathrm{sh}(\varepsilon_+)}{\mathrm{sh}(\alpha_a(s))\varepsilon_+}.$$

We can use the inversion method described in the Appendix and get the density. This is done in Figure 18, the parameters used are given in Table 3.



(a) $\lambda_0 = \mu_0 = 1$, $\lambda_+ = 0.4$, $\mu_+ = 0.6$, $\lambda_- = 0.1$, $\mu_- = 0.9$, $a = 3$ and $b = 5$



(b) $\lambda_0 = \mu_0 = 1$, $\lambda_+ = 0.9$, $\mu_+ = 0.1$, $\lambda_- = 0.6$, $\mu_- = 0.4$, $a = 5$ and $b = 3$

Figure 16: The density for the time a continuous time Markov chain spends above the x-axis. The point mass in zero is the probability that the Markov chain never enters a positive state.

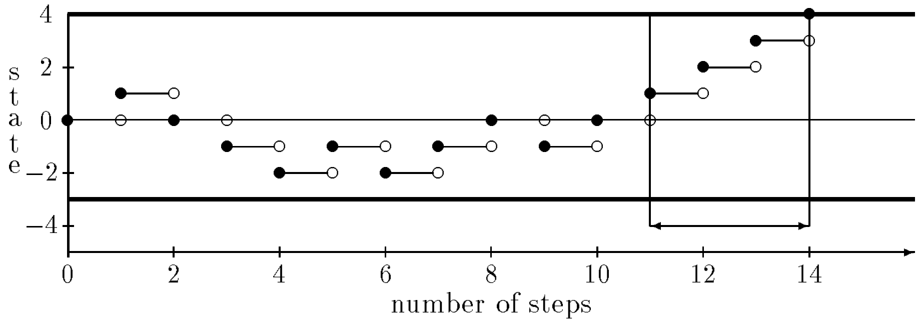


Figure 17: The time from when the process leaves the x-axis for the last time until absorption.

r_a	ν_+	σ_+^2	a	p_+	c	l_+	Figure
1	1	0.01^2	101	0.995	0.0099	$9.8 * 10^{-3}$	18(a)
1	1	0.01^2	1000	0.55	0.001	$1.0 * 10^{-4}$	18(b)
1	-1	1	5	0.4	0.2	$4.0 * 10^{-2}$	18(c)
1	-1	1	20	0.475	0.05	$2.5 * 10^{-3}$	18(d)
1	0	3^2	5	0.5	0.2	$4.4 * 10^{-3}$	18(e)
1	0	3^2	100	0.5	0.01	$1.1 * 10^{-5}$	18(f)

Table 3: Parameters in Figure 18.

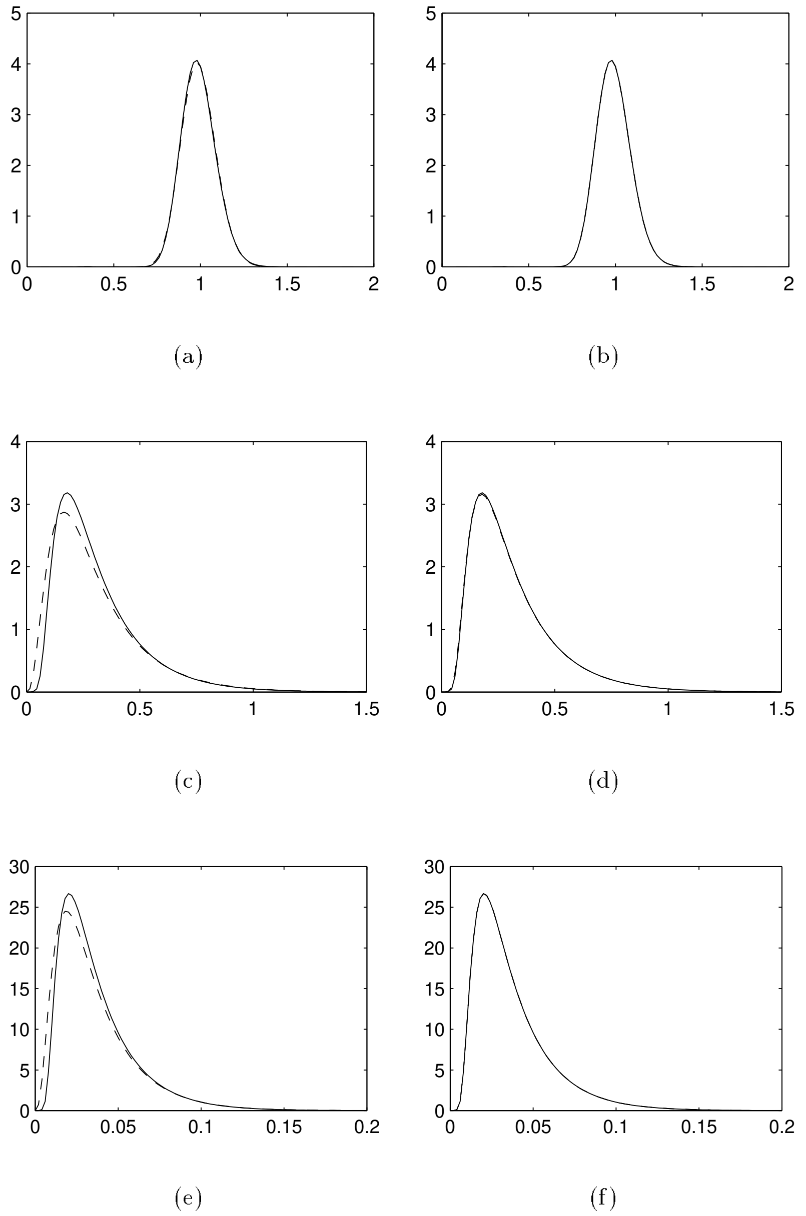


Figure 18: The density for the time when the process leaves the x-axis for the last time until absorption in a, conditioned on absorption in a. The solid line is the limit density and the dashed line represents the scaled continuous time Markov chain, the parameters used are given in Table 3.

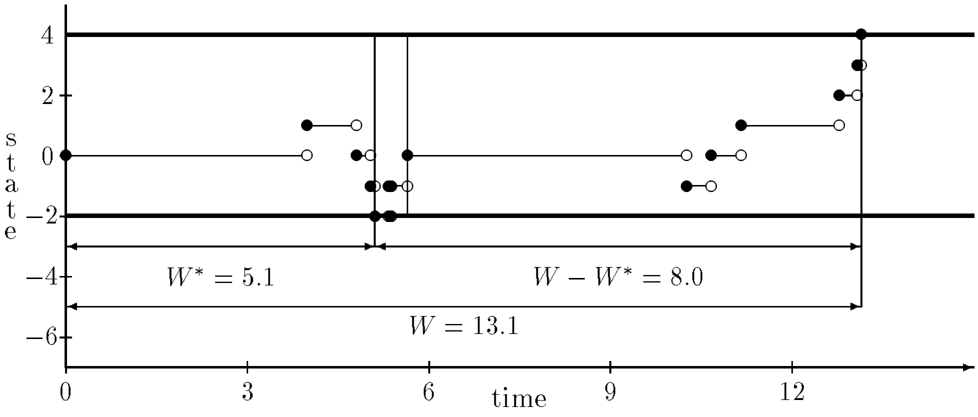


Figure 19: Realization of a cover time with range $m = 6$ of a continuous time Markov chain with state depending transition probabilities. Here $S_W = 4$ and $S_{W^*} = -2$.

5.4 Cover time of range m

The cover time of range m is defined as

$$W \stackrel{\text{def}}{=} \inf\{t \geq 0 : \max_{0 \leq s \leq t} S_s - \min_{0 \leq s \leq t} S_s = m\}$$

i.e. the minimum time it takes for the random process $\{S_t\}_{t \geq 0}$ to visit m different states, except 0. The endpoint, S_W, belongs to $[-m, m]$ hence we consider

$$E\left[e^{-sW}\right] = \sum_{k=1}^{m} \left(E\left[e^{-sW} I_{S_W=k}\right] + E\left[e^{-sW} I_{S_W=-k}\right] \right).$$

When the end point is given the process can be split into three disjoint independent parts. Let W^* be the first time the opposite endpoint is reached, i.e. $|S_W - S_{W^*}| = m$, see Figure 19, defined formally as

$$W^* \stackrel{\text{def}}{=} \inf\{t \geq 0 : S_t = \operatorname{sign}(S_W)(|S_W| - m)\}.$$

The first part lasts from the start to the time W^*, the second part lasts from the time W^* until the first time the process hits 0 after W^* and the third part starts where the second part ends and ends where the process is stopped, see Figure 19. On each of these three parts the Laplace transform for the first passage time, (20), can be applied. For $1 \leq k \leq m-1$

$$
\begin{aligned}
E\left[e^{-sW} I_{S_W=k}\right] &= E\left[e^{-sW^*} e^{-s(W-W^*)} I_{S_W=k}\right] \\
&= \frac{\frac{q_0}{1+sl_0} e^{-(m-k-1)\zeta_-} \frac{\operatorname{sh}(\chi_-(s))}{\operatorname{sh}((m-k)\chi_-(s))}}{1-\frac{1}{1+sl_0}\left(p_0 e^{-\zeta_+}\frac{\operatorname{sh}((k-1)\chi_+(s))}{\operatorname{sh}(k\chi_+(s))} + q_0 e^{\zeta_-}\frac{\operatorname{sh}((m-k-1)\chi_-(s))}{\operatorname{sh}((m-k)\chi_-(s))}\right)} \\
&\qquad \times \frac{\frac{p_-}{1+sl_-} e^{(m-k-1)\zeta_-} \frac{\operatorname{sh}(\chi_-(s))}{\operatorname{sh}((m-k)\chi_-(s))}}{1-\frac{p_-}{1+sl_-} e^{-\zeta_-}\frac{\operatorname{sh}((m-k-1)\chi_-(s))}{\operatorname{sh}((m-k)\chi_-(s))}} \\
&\qquad \times \frac{\frac{p_0}{1+sl_0} e^{(k-1)\zeta_+} \frac{\operatorname{sh}(\chi_+(s))}{\operatorname{sh}(k\chi_+(s))}}{1-\frac{1}{1+sl_0}\left(p_0 e^{-\zeta_+}\frac{\operatorname{sh}((k-1)\chi_+(s))}{\operatorname{sh}(k\chi_+(s))} + q_0 e^{\zeta_-}\frac{\operatorname{sh}((m-k)\chi_-(s))}{\operatorname{sh}((m-k+1)\chi_-(s))}\right)}
\end{aligned}
$$

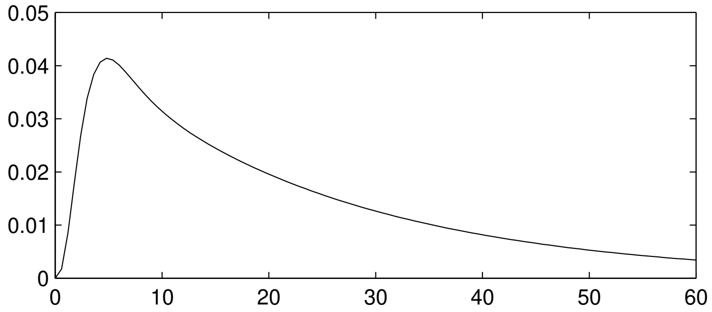
and similarly for $-(m-1) \leq k \leq -1$. For $k=m$ and $k=-m$ it is enough to consider the third term:

$$
E\left[e^{-sW} I_{S_W=m}\right] = \frac{\frac{p_0}{1+sl_0} e^{(m-1)\zeta_+} \frac{\operatorname{sh}(\chi_+(s))}{\operatorname{sh}(m\chi_+(s))}}{1-\frac{1}{1+sl_0} p_0 e^{-\zeta_+}\frac{\operatorname{sh}((m-1)\chi_+(s))}{\operatorname{sh}(m\chi_+(s))}}
$$

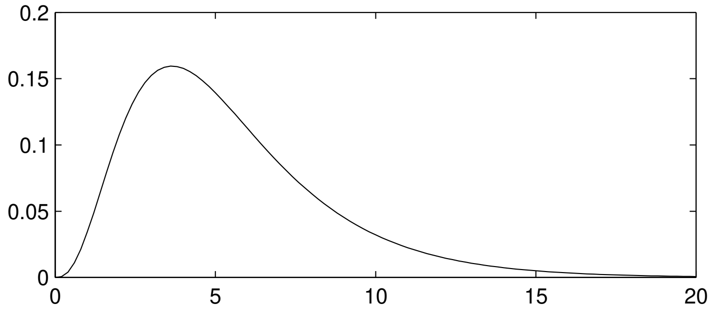
and similarly for $k=-m$. Adding the formulas above we get the Laplace transform for W:

$$
\begin{aligned}
E\left[e^{-sW}\right] = \sum_{k=1}^{m-1} \frac{q_0 p_0}{(1+sl_0)^2} \mathrm{sh}(\chi_+(s))\mathrm{sh}(\chi_-(s)) \\
\times \Bigg\{ \frac{q_+ e^{-(k-1)\zeta_-} \frac{\mathrm{sh}(\chi_+(s))}{\mathrm{sh}^2((m-k)\chi_+(s))\mathrm{sh}(k\chi_-(s))}}{1 - \frac{1}{1+sl_0}\left(p_0 e^{-\zeta_+} \frac{\mathrm{sh}((m-k-1)\chi_+(s))}{\mathrm{sh}((m-k)\chi_+(s))} + q_0 e^{\zeta_-} \frac{\mathrm{sh}((k-1)\chi_-(s))}{\mathrm{sh}(k\chi_-(s))}\right)} \\
\times \frac{\frac{1}{1+sl_+}}{1 - \frac{q_+}{1+sl_+} e^{\zeta_+} \frac{\mathrm{sh}((m-k-1)\chi_+(s))}{\mathrm{sh}((m-k)\chi_+(s))}} \\
\times \frac{1}{1 - \frac{1}{1+sl_0}\left(p_0 e^{-\zeta_+} \frac{\mathrm{sh}((m-k)\chi_+(s))}{\mathrm{sh}((m-k+1)\chi_+(s))} + q_0 e^{\zeta_-} \frac{\mathrm{sh}((k-1)\chi_-(s))}{\mathrm{sh}(k\chi_-(s))}\right)} \\
+ \frac{p_- e^{(k-1)\zeta_+} \frac{\mathrm{sh}(\chi_-(s))}{\mathrm{sh}^2((m-k)\chi_-(s))\mathrm{sh}(k\chi_+(s))}}{1 - \frac{1}{1+sl_0}\left(p_0 e^{-\zeta_+} \frac{\mathrm{sh}((k-1)\chi_+(s))}{\mathrm{sh}(k\chi_+(s))} + q_0 e^{\zeta_-} \frac{\mathrm{sh}((m-k-1)\chi_-(s))}{\mathrm{sh}((m-k)\chi_-(s))}\right)} \\
\times \frac{\frac{1}{1+sl_-}}{1 - \frac{p_-}{1+sl_-} e^{-\zeta_-} \frac{\mathrm{sh}((m-k-1)\chi_-(s))}{\mathrm{sh}((m-k)\chi_-(s))}} \\
\times \frac{1}{1 - \frac{1}{1+sl_0}\left(p_0 e^{-\zeta_+} \frac{\mathrm{sh}((k-1)\chi_+(s))}{\mathrm{sh}(k\chi_+(s))} + q_0 e^{\zeta_-} \frac{\mathrm{sh}((m-k)\chi_-(s))}{\mathrm{sh}((m-k+1)\chi_-(s))}\right)} \Bigg\} \\
+ \frac{\frac{p_0}{1+sl_0} e^{(m-1)\zeta_+} \frac{\mathrm{sh}(\chi_+(s))}{\mathrm{sh}(m\chi_+(s))}}{1 - \frac{1}{1+sl_0} p_0 e^{-\zeta_+} \frac{\mathrm{sh}((m-1)\chi_+(s))}{\mathrm{sh}(m\chi_+(s))}} + \frac{\frac{q_0}{1+sl_0} e^{-(m-1)\zeta_-} \frac{\mathrm{sh}(\chi_-(s))}{\mathrm{sh}(m\chi_-(s))}}{1 - \frac{1}{1+sl_0} q_0 e^{\zeta_-} \frac{\mathrm{sh}((m-1)\chi_-(s))}{\mathrm{sh}(m\chi_-(s))}}.
\end{aligned}
$$

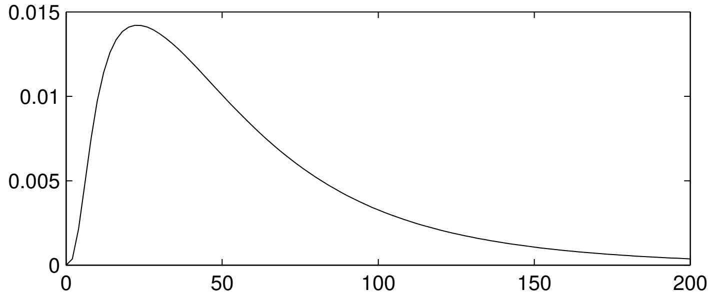
This complicated Laplace transform can be inverted numerically by the method described in the Appendix. In Figure 20 three examples are plotted.



(a) $p_0 = 0.1$, $p_+ = 0.9$, $p_- = 0.9$ and $m = 4$



(b) $p_0 = 0.5$, $p_+ = 0.8$, $p_- = 0.2$ and $m = 4$



(c) $p_0 = 0.5$, $p_+ = 0.2$, $p_- = 0.8$ and $m = 5$

Figure 20: The density of the cover time of range m for a continuous time Markov chain with state depending transition probabilities. The mean holding times are 1. The tails are cut off.

Appendix

In this paper we have used a method proposed by Abate and Whitt [2] called Euler's method to invert Laplace transforms. In [2] Abate and Whitt presents two different methods for inverting Laplace transforms numerically. The first is called Euler's method (because Euler summation is used) and the second is called Post-Widder (because it is based on a Theorem called Post-Widder Theorem). Euler's method is a variant of the Fourier-series method [1] which was first proposed by Dubner and Abate in 1968 [5].

To invert the Laplace transform they use the inversion formula

$$\begin{aligned}
f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}\widehat{f}(s)ds &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(a+iu)t}\widehat{f}(a+iu)du \\
&= \frac{e^{at}}{2\pi}\int_{-\infty}^{\infty}(\cos(ut)+i\sin(ut))\widehat{f}(a+iu)du \\
&= \frac{e^{at}}{2\pi}\int_{-\infty}^{\infty}[\Re(\widehat{f}(a+iu))\cos(ut)-\Im(\widehat{f}(a+iu))\sin(ut)]du \\
&= \frac{2e^{at}}{\pi}\int_{0}^{\infty}\Re(\widehat{f}(a+iu))\cos(ut)du \qquad (32)
\end{aligned}$$

where $\widehat{f}(s)$ is the Laplace transform and a is any vertical line such that the Laplace transform has no singularities to the right of it. To approximate the integral, (32), numerically it is discretisized by the trapezoidal rule with step size h:

$$f(t) \approx f_h(t) \equiv \frac{he^{at}}{\pi}\Re(\widehat{f}(a)) + \frac{2he^{at}}{\pi}\sum_{k=1}^{\infty}\Re(\widehat{f}(a+ikh))\cos(kht)$$

let $h = \frac{\pi}{2t}$ and $a = \frac{A}{2t}$ then

$$= \frac{e^{A/2}}{2t}\Re\Big(\widehat{f}\Big(\frac{A}{2t}\Big)\Big) + \frac{e^{A/2}}{t}\sum_{k=1}^{\infty}(-1)^k\Re\Big(\widehat{f}\Big(\frac{A+2k\pi i}{2t}\Big)\Big). \qquad (33)$$

The the trapezoidal rule causes a discretization error. To analyze the error the Poisson summation formula ([6] page 111) is used. The essential idea is to replace the damped density, $e^{-bt}f(t)$, with a periodic function, $f_p(t)$, that can be represented by its Fourier-series i.e.

$$f_p(t) = \sum_{k=-\infty}^{\infty} e^{-b\left(t+\frac{2\pi k}{h}\right)}f\Big(t+\frac{2\pi k}{h}\Big) = e^{-bt}f(t) + \sum_{k=1}^{\infty} e^{-b\left(t+\frac{2\pi k}{h}\right)}f\Big(t+\frac{2\pi k}{h}\Big)$$

with period $2\pi/h$. When $h = \frac{\pi}{t}$ and $b = \frac{A}{2t}$

$$f(t) = e^{\frac{A}{2}} f_p(t) - \sum_{k=1}^{\infty} e^{-Ak} f(t(2k+1)). \tag{34}$$

The periodic function can be written as a complex Fourier-series

$$f_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikht}$$

where c_k is the k$^{\text{th}}$ Fourier coefficient of $f_p(t)$ which turns out to be

$$\frac{h}{2\pi}\widehat{f}(b+ikh) = \frac{1}{2t}\widehat{f}\Big(\frac{A+2k\pi i}{2t}\Big).$$

Inserting the periodic function $f_p(t)$ in (34) we get

$$f(t) = \frac{e^{A/2}}{2t} \sum_{k=-\infty}^{\infty} (-1)^k \Re\Big(\widehat{f}\Big(\frac{A+2k\pi i}{2t}\Big)\Big) - \sum_{k=1}^{\infty} e^{-kA} f((2k+1)t).$$

The first sum coincides with the approximation, (33), which yields that the last term is the discretization error. The error is small when A is large but if A is too large it will lead to computation difficulties because of the term $e^{A/2}$. The remaining problem is to calculate (33) numerically which contains an infinite sum. When $\Re\big(\widehat{f}\big(\frac{A+2k\pi i}{2t}\big)\big)$ have a constant sign (33) will be an alternating sum. For alternating sums there are methods to accelerate the convergence, here Euler's method ([6] page 112) is used. Let $s_n(t)$ be the truncated sum after n terms in (33) i.e.

$$s_n(t) = \frac{e^{A/2}}{2t}\Re\Big(\widehat{f}\Big(\frac{A}{2t}\Big)\Big) + \frac{e^{A/2}}{t}\sum_{k=1}^{n}(-1)^k \Re\Big(\widehat{f}\Big(\frac{A+2k\pi i}{2t}\Big)\Big).$$

After the first n terms, apply Euler summation to $m+1$ terms i.e.

$$E(m,n,t) = \sum_{k=0}^{m} \binom{m}{k} s^{-m} s_{n+k}(t).$$

Which yields

$$f(t) \approx s_n(t) + E(m,n,t).$$

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