KTH Mathematics
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Exercises in Financial Mathematics

1. Risk adjusted probability measures

1. We consider a stochastic payment $X(\omega)$ which pays out a dividend according to the following table

ω_1	ω_2	ω_3	price
5	10	15	11.75
20	0	10	8.00

Here the ω_1, ω_2, ω_3 are three possible cases, and the price given on the right is the forward price.

a) Determine the risk adjusted probabilities ($q_i = p_i U(\omega_i)$) of the three cases.

b) Determine the forward price G_0 of the contract

ω_1	ω_2	ω_3	price
50	100	30	G_0

Answers: 1a) $[q_1, q_2, q_3] = [0.15, 0.35, 0.50]$ b) 57.50 dollar.

2. Conditional expectations

1. An urn contains five envelopes: two blue and three red. The blue contain respectively 50 dollars and 150 dollars, whilst the red contain respectively 100, 200 and 500 dollars. We randomly draw an envelope out of the urn.

   a) Determine the expected value of the contents of the envelope.

   b) Determine the conditional expected value of the contents of an envelope given that it is blue.

   c) Determine the conditional expected value of the contents of an envelope given that it is red.

   We can now regard the conditional expected values of the contents as a random variable $X = X(\omega)$ on the sample space {blue, red}:
   $X(\omega) = \mathrm{E}[\text{contents} \mid \text{colour of envelope is } \omega]$

   d) What is $X(\omega)$ for each of the two values {blue, red} on ω?

   e) Compute $\mathrm{E}[X]$ by regarding the random variable $X(\omega)$. Compare the answer to the answer in a).

2. Let us consider the following game: A player tosses a coin four times. There are sixteen possible outcomes: (heads, heads, heads, heads), (heads, heads, heads, tails) etc. One gets 10 dollars every time heads turns up, whereas tails does not pay at all. Also, if heads turns up all four times, one gets 100 dollars.

Consider the situation after two toss ups: The sample space consists of four possible cases:

$$\{\omega_1, \omega_2, \omega_3, \omega_4\}$$
$$= \{\text{(heads, heads), (heads, tails), (tails, heads), (tails, tails)}\}.$$

Let X be the random variable "Money won after four toss ups" and determine the conditional expected value after two toss ups $\mathrm{E}[X \mid \omega]$, where ω can take the four values $\omega_1, \ldots, \omega_4$.

3. We choose a point x randomly in the interval $[0, 1]$ (with uniform distribution.) Then we choose a point y randomly (with uniform distribution) in the interval $[0, x]$.

   a) Determine the conditional expected value of y given x: $\mathrm{E}[y \mid x]$.

   b) Determine the expected value of y: $\mathrm{E}[y]$.

Answers: 1a) 200 dollars b) 100 dollars c) $266\,\frac{2}{3}$ dollars
2) $\mathrm{E}[X \mid \omega_1] = 45$, $\mathrm{E}[X \mid \omega_2] = 20$, $\mathrm{E}[X \mid \omega_3] = 20$, $\mathrm{E}[X \mid \omega_4] = 10$
3a) $\frac{1}{2}x$ 3b) $\frac{1}{4}$

3. Forwards and Futures

Here we assume interest is continuously compounded.

1. A share is valued at present at 80 dollars. In nine months it will give a dividend of 3 dollars. Determine the forward price for delivery in one year given that the rate of interest is 5% a year.

2. A share is valued at present at 80 dollars. In nine months it will give a dividend of 4% of its value at that time. Determine the forward price for delivery in one year given that the rate of interest is 5% a year.

3. The current forward price of a share to be delivered in one year is 110 dollars. In four months the share will give a dividend of 2 dollars and in ten months will give a dividend of 2% of its value at that time. Determine the current spot price of the share given that the rate of interest is 6% a year.

4. The exchange rate of US dollars is today 8.50 SEK per dollar. The forward price of a dollar to be delivered in six months is 8.40 SEK. If the Swedish six month interest rate is 4% a year, determine the American six month interest rate.

5. The forward price of a US dollar the first of August with delivery at the end of December is 0.94630 Euros. The forward price of a dollar to be delivered at the end of June next year is 0.95152 Euros. Assuming a flat term structure for both countries and that the Euro interest rate is 4% a year—what is the American rate of interest?

6. Determine the forward price of a bond to be delivered in two years. The bond pays out 2 Euros every 6-months during $4\frac{1}{2}$ years (starting in six months), and 102 Euros after five years. Thus the bond is, as of today, a 5-year 4%-coupon bond with a coupon dividend every six months with a 100 Euro face value.

   The bond is to be delivered in two years immediately after the dividend has been paid. The present term structure is given by the following rates of interest (on a yearly basis)

6 months	5.0%	18, 24 months	5.6%
12 months	5.4%	30–60 months	5.9%

7. A one-year forward contract of a share which pays no dividend before the contract matures is written when the share has a price of 40 dollars and the risk-free interest rate is 10% a year.

   a) What is the forward price?

   b) If the share is worth 45 dollars six months latter, what is the value of the original forward contract at this time? If another forward contract is to be written with the same date of maturity, what should the forward price be?

Answers: 1) 81.06 2) 80.74 3) 107.67 4) 6.37% 5) 2.90% 6) 94.05 7a) 44.207 7b) 2.949 dollars, 47.307 dollars.

4. Interest Rates and Duration

1. Calculate approximately the duration of a portfolio containing a coupon bearing-bond which matures in two years with face value 100,000 SEK and pays a 6%-coupon (this means that the coupon is paid every six month at 3% of the face value,) plus a short position of a futures contract with maturity in two years on a three year (at the time of maturity of the futures) 6% coupon-bearing bond (the first coupon payment is six months after the maturity of the futures) with face value 50,000 SEK. Interest rates are today 5.5% a year with continuous compounding for any length of duration. (Approximate the futures price with the forward price.)

2. We consider an interest rate swap where one party at the end of year 1, 2, 3 and 4 gets the floating one-year rate of interest on a nominal principal of 100,000 SEK while he pays a sum c at the end of each year. The zero-coupon interest rates for 1-, 2-, 3- and 4-year durations are respectively 10, 11, 12 and 13% a year, with continuous compounding.

   a) Determine the value of the swap for the party that gets the floating interest rate if $c = 11{,}000$ SEK.

   b) Determine c so that the value of the swap when it is contracted is zero.

3. Determine the
   a) forward price
   b) forward yield
   c) forward duration

   two years into the future for a bond that pays out 100 SEK in 2.5, 3 and 3.5 years and 2 100 SEK in 4 years. Zero-coupon interest rates are at present 6%, 6.5%, 7%, 7.5% and 8% for the duration of 2, 2.5, 3, 3.5 and 4 years.

Answers: 1) 0.514 years 2a) 7 552.86 SEK 2b) 13,518.00 SEK
3a) 1993.29 SEK 3b) 9.939% 3c) 1.861 years.

5. **European options** and other derivatives

Here we assume interest is continuously compounded.

1. Determine the price of a European futures call option on a barrel of crude oil to be delivered in 4 months. This is also the time of maturity of the option. The futures price today is $F_0 = \$25.00$, the strike price is \$23.00 and the volatility of the futures price is estimated to be 25% for one year. The risk-free interest rate is 9% per year.

2. Determine the price of a European call option on a share which does not pay dividends before the maturity of the option. The present price of the share is 45 SEK, the option matures in 4 months, the volatitlity is 25% in one year, the option's strike price is 43 SEK and the risk-free interest rate is 9% a year.

3. The same question as above, but now we assume the share pays a dividend of 0.50 SEK in 3 months, all other assumptions are the same.

4. Determine the price of a European put option on 1 GBP with a strike price of 14 SEK in 6 months. The exchange rate is 13 SEK for 1 GBP, and the pound's volatility is assumed to be 14% in one year. The pound's rate of interest is 11% and the Swedish crown's rate of interest is 7% a year.

5. Determine the price of a European call option on an index of shares which is expected to give a dividend of 3% a year, continuously. The current value of the index is 93 SEK, the strike price is 90 SEK and the option matures in two months. The risk-free interest rate is 8% a year and the index has a volatility of 20% in one year.

6. Let $S(t)$ be the spot price of a share at time t (year) which does not pay dividends the following two years. Determine the price of a contract which after two years gives the owner $S(0)\dfrac{S(2)}{S(1)}$ SEK. The risk-free rate of interest is 6% a year.

7. Let $S(t)$ be the spot price of a share at time t (year) which does not pay dividends the following year. Determine the price of a contract which after one year gives the owner $\dfrac{S(1)^2}{S(0)}$ SEK. The risk-free rate of interest is 6% a year, and the share's volatility is assumed to be 30% for one year.

8. Determine the price of a European put option with maturity in two years on a five year 6% coupon bond issued today (the first coupon payment after the maturity of the option occurs six months after the maturity of the option) with face value 50 000 SEK, which is also the option's strike price. The forward yield of the bond is 6.5% a year and the standard deviation of the yield is 0.012 in one year. The two year zero-coupon rate of interest is 5.5% a year.

9. Calculate the value of a one-year put option on a ten year bond issued today. Assume that the present price of the bond is 1,250 SEK, the strike price of the option is 1,200 SEK, the one-year interest rate is 10% per year, the forward yield has a standard deviation of 0.013 in one year, the forward duration of the bond as of the time of maturity for the option is 6.00 years and the present value of the coupon payments which will be paid out during the lifetime of the option is 133 SEK.

Answers: 1) 2.527 2) 4.463 3) 4.115 4) 1.331 SEK 5) 5.183 SEK
6) $S(0)e^{-0.06}$ 7) $S(0)e^{0.15}$ 8) 1,253 SEK 9) 20.90 SEK

6. Binomial Trees

1. The futures price of coffee (20 kg) to be delivered in four weeks is 100 SEK. The volatility is 2% in a week. The risk-free rate of interest is 0.1% per week. In the following cases, use a binomial tree with time interval of one week.

   a) Determine the price of a European futures put option on coffee (20 kg) with strike price 104 SEK. The option matures in four weeks (which is also the maturity of the futures.)

   b) The same question, though for an American option, ceteris paribus (latin: "everything else being the same").

2. A share costs today 99.6008 SEK. The volatility of the futures price is 2% in a week, and the share pays no dividends the coming month. The risk-free rate of interest is 0.1% a week. In the following cases, use a binomial tree with time interval one week.

   a) Determine the price of an American put option with maturity in four weeks at a strike price of 104 SEK.

   b) The same question, but the share pays a dividend of 2 SEK in just under two weeks (that is just before time 2 in the binomial tree.) Also, the present price of the share is 101.5968 SEK, ceteris paribus.

3. Determine the price of an American option to buy 100 GBP at 13 SEK per pound, which is the value of the pound today. The time to maturity is one year, the pound's volatility relative to the Swedish crown is 10% in one year. The rate of interest of a pound is 8% a year, whereas the interest rate of the Swedish crown is 4% a year. Use a binomial tree to solve the problem, with time interval tree months.

4. Determine the price of an American call option on a share whose present price is 100 SEK with maturity in one year with strike price 98 SEK. The share's volatility (or rather the forward price of the share) is 20% in one year, the rate of interest is 6% a year, and the share will pay a dividend in 2.5 months of 4% of the value of the share at that time. Use a binomial tree with time interval three months.

5. Calculate the price of a four month put option on a share which pays no dividend and whose present price is 60 SEK. The strike price is 62.50 SEK, the risk-free interest rate 10% a year and the volatility 45% in one year. Use a binomial tree with time interval one month.

6. Calculate the price of an eight month American call option on a maize futures when the present futures price is 198 SEK, the strike price of the option is 200 SEK, the risk-free rate of interest is 8% and the volatility of the futures price is 30% over one year. Use a binomial tree with time interval two months.

7. A two month American put option on an index of shares has the strike price 480. The present value of the index is 484, the risk-free rate of interest is 10% per year, the dividend of the index is 3% per year and the volatility is 25% over a year. Determine the value of the option by using a binomial tree with time interval of half a month.

8. The spot price of copper is 60 SEK per decaskålpund. Assume that, at present, the futures price of copper is

maturity	futures price
3 months	59
6 months	57
9 months	54
12 months	50

The volatility of the price of copper is 40% during one year and the risk-free interest rate is 6% a year. Use a suitable binomial tree with time interval three months to estimate the price of an American call option on copper with strike price 60 SEK and maturity in one year.

Answers: 1a) 4.2481 SEK. 1b) 4.2593 SEK.
2a) 4.4685 SEK. 2b) 4.4166 SEK.
3) 34.50 SEK. 4) 9.3451 SEK. 5) 6.8435 SEK. 6) 17.6439 SEK.
7) 15.2336 8) 6.3167 SEK.

7. Interest Rate Derivatives (Ho-Lee)

1. The following zero-coupon rates of interest (with continuous compounding) holds: 2-year: 1-year: 8%, 2-year: 8.25%, 3-year: 8.5%, 4-year: 8.75%. The volatility of the one-year rate is assumed to be 1.5% during one year. Determine the price of a European call option with maturity in two years on a (at maturity of the option) zero coupon bond with face value 100 SEK (thus the bond matures in four years time from the present.) The strike price of the option is 80 SEK.
   a) Use a binomial tree based on Ho-Lee's model with a time interval of one year.
   b) Calculate the value using Ho-Lees model when the increments have a Normal distribution (i.e., Black's model.)

2. Calculate the value of a callable bond maturity in 10 years and strike price price 100. The bond can be exercised after 3 years at 70 SEK, after 6 years at 80 SEK and after 8 years at 90 SEK. The volatility of the one-year rate of interest is assumed to be 1.5% during one year. The present zero coupon rates of interest are (% per year with continuous compounding):

maturity	interest	maturity	interest
1 year	4.0	6 year	5.0
2 year	4.2	7 year	5.2
3 year	4.4	8 year	5.4
4 year	4.6	9 year	5.6
5 year	4.8	10 year	5.8

   Use a binomial tree with time interval of one year. (We suggest this is done using a spreadsheet.)

3. Calculate the futures price of a ten-year zero-coupon bond with face value 100 and maturity in six years. The volatility of the one year rate of interest is assumed to be 1.5% during one year, and the zero-coupon rate of interest is the same as in the previous question.
   a) Use a binomial tree with time interval of one year based on Ho-Lees model. (We suggest you do this using a spreadsheet.)
   b) Do the calculation analytically using Ho-Lee's model when the increments have a Normal distribution.

Answers: 1a) 3.0205 SEK. 1b) 2.9324 SEK. 2) 53.1950 3a) 74.5689 3b) 74.3639. Forward price = 75.5784.