Critical point configurations of the 5-, 6-, 7-, and 8-point algorithms for relative orientation.

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Abstract: It is well known that certain object point configurations are dangerous in relative orientation. We digress on this and consider in detail the number of points that are allowed to belong to a dangerous set and how many other points that are needed for solvability. We show how the various algorithms can detect when a point configuration is critical, i.e. doesn't give a unique solution.

1. Introduction

The probably best known critical configuration in aerial photogrammetry is the dangerous cylinder. If all ground points sit on a cylinder, it may be impossible to determine the relative orientation of the aeroplane even if a large number of ground points are registered. This cylinder is a special case of a family of quadric surfaces that are called dangerous. See references [5], [7], [11], [12]. In section 4, we shall give a short account of the idea put forward in these papers.

In this paper, we shall study what happens if some points sit on a dangerous surface and some don't. We shall give conditions for the solvability of the relative orientation problem for various number of points on and off different dangerous sets. We shall call a solution unique if it is unique up to a scale factor for the translation of the camera.

We define a point configuration in 3D as critical *for an algorithm if the algorithm fails to produce a unique solution from the perspective coordinates of that point configuration.* The algorithms, shortly described in section 3, will be able to detect criticality by using test variables that become zero if the data are critical.

We shall consider exact equations for the critical sets so this paper is primarily about exact data without noise. With noisy data, the test variables that signal criticality cannot be expected to be exactly zero but must be allowed to belong to a confidence interval. If a test variable belongs to such an interval, the calculated parameters will be very inaccurate. When more than eight, perhaps noisy, data are present the 8-point algorithm is used and a least squares fitting procedure is employed in the first step of the algorithm.

We have proven in [9] that the relative orientation (also called motion) problem has a unique solution when at least six generic point pairs are known. For five point pairs, there is in general no unique solution and one cannot speak about a critical configuration for the five point problem. There are, however, configurations of five points that defeat the

five point algorithm.
Because the algorithms for fewer points are more elaborate than those for more points, they are of interest even if many points are known. For instance, a configuration consisting of 7 points in a plane and one point outside the plane will defeat the 8-point algorithm. Disregarding one point in the plane and feeding the remaining points to the 7-point algorithm will produce a unique solution. In the same way, the 5-point algorithm can be used for configurations that are critical for the 6-, 7-, and 8-point algorithms. It will then produce several solutions, one of which may be selected by some extra information.
The mathematical formulation of the problem is given in sections 2 and 3. The equation of a dangerous surface is deduced in section 4 and two critical point confugurations are described in section 5. A stable way of handling the equations numerically is presented in sections 6 and 7. The main results about the detection of different critical configurations by the algorithms are presented in section 8. We suggest a way of handling critical configurations in section 9 and consider noisy data in section 10. Two special questions about the criticality of an arbitrary plane and cylinder are answered in two appendices.

2. Notations and mathematical formulation.

The coordinates of points in R^3 will be denoted $\boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$. The origin will be the center of perspectivity and $X_3 = 1$ will be the embedded projective plane so that the perspective image of $\boldsymbol{X}$ is $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} X_1/X_3 \\ X_2/X_3 \\ 1 \end{bmatrix}$.
Our problem is to determine the motion of a rigid object characterized by n points. Let $\boldsymbol{X}^i$, $(1 \le i \le n)$ be the original positions of the n points and let their corresponding positions after the motion be $\boldsymbol{Y}^i$, $(1 \le i \le n)$. Any motion can be described by a rotation matrix $\boldsymbol{R}$ and a translation vector $\boldsymbol{t}$

$$\boldsymbol{Y}^i = \boldsymbol{R}\boldsymbol{X}^i + \boldsymbol{t} . \tag{2.1}$$

We have chosen this formulation, common in computer vision, where the camera is assumed to be fixed at the origin and the object is moving. In relative orientation, the camera is moving and the object is fixed. Then, the equation has the form $\boldsymbol{X} = \boldsymbol{R}\boldsymbol{Y} + \boldsymbol{b}$ which is equivalent to (2.1).

The objective is to determine $\boldsymbol{R}$ and $\boldsymbol{t}$ from the pairs of perspective coordinates $(\boldsymbol{x}^i, \boldsymbol{y}^i)$. Since the distance to the object is not known, the length of $\boldsymbol{t}$ cannot be determined so the sought motion parameters are the three parameters describing $\boldsymbol{R}$ and the two parameters describing the direction of $\boldsymbol{t}$. Two determine these five parameters, at least five point pairs $(\boldsymbol{x}^i, \boldsymbol{y}^i)$ must be known.

The determination of $\boldsymbol{R}$ and $\boldsymbol{t}$ is based on the following geometrical considerations.

By (2.1), the vectors $\boldsymbol{Y}^i$, $\boldsymbol{R}\boldsymbol{X}^i$, and $\boldsymbol{t}$ are in the same plane and are said to satisfy the *coplanarity condition*. This implies that the determinant formed of the three vectors is zero. This can also be written

$$\boldsymbol{Y}^{i^t}\boldsymbol{C}\boldsymbol{R}\boldsymbol{X}^i = 0, \tag{2.2}$$

where

$$\boldsymbol{C} = \begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix} \tag{2.3}$$

is the so called *axiator* matrix performing cross product with $\boldsymbol{t}$. The *essential matrix* $\boldsymbol{E}$ is defined by

$$\boldsymbol{E} = \boldsymbol{C}\boldsymbol{R}. \tag{2.4}$$

After division by $\boldsymbol{X}^i{}_3\boldsymbol{Y}^i{}_3$ in (2.2), we obtain from each point pair the following homogeneous linear equation for the elements of $\boldsymbol{E}$

$$\boldsymbol{y}^{i^t}\boldsymbol{E}\boldsymbol{x}^i = 0 . \tag{2.5}$$

Knowing eight point pairs $(\boldsymbol{x}^i, \boldsymbol{y}^i)$, one can in general determine the relative size of the nine elements of $\boldsymbol{E}$ from (2.5). This calculation forms the first part of the 8-point algorithm, [10]. A unique solution is obtained by normalizing $\boldsymbol{E}$ by

$$||\boldsymbol{E}||_F = 1 , \tag{2.6}$$

where $||\ ||_F$ stands for the Frobenius matrix norm.

Contrary to the common practise of normalizing $\boldsymbol{E}$ by putting $e_{33} = 1$, this normalization has the advantage of being applicable also when $e_{33} = 0$.

With only five, six, or seven point pairs, some properties of $\boldsymbol{E}$ must be supplemented and this is done by the respective algorithms.

The topic of this paper is the instance that the determination of $\boldsymbol{E}$ fails because the point pairs have a particular configuration.
If $\boldsymbol{E}$ is found, two $\boldsymbol{R}$ and two $\boldsymbol{C}$ can be determined that satisfy (2.4). A unique pair $\boldsymbol{R}$, $\boldsymbol{C}$ can be determined by using the fact that the object is in front of the camera both before and after the motion, i.e $X^i{}_3 > 0$, $Y^i{}_3 > 0$, see [11].

3. The 5-, 6-, anf 7-point algorithms.

As mentioned in connection with equation (2.5), eight point pairs are sufficient for determining the relative size of the nine elements of $\boldsymbol{E}$. When fewer than eight point pairs are at hand, one must use the special property of $\boldsymbol{E}$ of being the product $\boldsymbol{E} = \boldsymbol{CR}$ of a skew-symmetric and a rotation matrix. This property can be formulated algebraically as (see [3] or [9]).

$$\boldsymbol{E}\boldsymbol{E}^t\boldsymbol{E} = \frac{1}{2} trace\,(\boldsymbol{E}\boldsymbol{E}^t)\,\boldsymbol{E}\ . \tag{3.1}$$

Having $n < 8$ linear equations (2.5) for the nine elememnts e_{ij} of $\boldsymbol{E}$, one can write the solution with the aid of $9 - n$ parameters λ_k

$$e_{ij} = \sum_{k=1}^{9-n} \alpha_{ijk}\lambda_k\ . \tag{3.2}$$

(With $n = 8$ there is only one parameter determining the size of $\boldsymbol{E}$.)
Inserting (3.2) in (3.1), we obtain nine algebraic equation of degree 3 in the parameters λ_k. These equations are used for determining the relative size of the λ_k in the 5-, and 6-point algorithms given in [9]. An analogous 7-point algorithm is easily constructed along the same lines.
The numerical implementation of all these algorithms will be discussed in sections 6 and 7.

4. The equation of a dangerous surface

The idea behind the various constructions of a dangerous surface in [5], [7], [12], is to choose the points $\boldsymbol{X}^i$ in 3D so that the system of equations (2.5) and (2.6) doesn't have a unique solution. These constructions are independent of the number of points since they imply a condition for each point separately. All the given references consider eight or more points and they don't involve equation (3.1).
The construction is carried out by choosing two different motions, i.e. two different essen-

tial matrices, say one "true" $\boldsymbol{E}$ and one "other" $\tilde{\boldsymbol{E}}$ and choosing the points so that they satisfy both equations

$$\begin{aligned} \boldsymbol{Y}^t\boldsymbol{E}\boldsymbol{X} &= 0 \\ \boldsymbol{Y}^t\tilde{\boldsymbol{E}}\boldsymbol{X} &= 0 \end{aligned} \quad . \tag{4.1}$$

Obviously, no unique $\boldsymbol{E}$ can be solved from these equations or their perspective variants. By definition, $\boldsymbol{X}$ and $\boldsymbol{Y} = \boldsymbol{R}\boldsymbol{X} + \boldsymbol{t}$ satisfy the first equation of (4.1). Inserting these points in the second equation of (4.1) gives the equation for the set of critical points for the true motion

$$(\boldsymbol{R}\boldsymbol{X} + \boldsymbol{t})^t\tilde{\boldsymbol{E}}\boldsymbol{X} = \boldsymbol{X}^t\boldsymbol{R}^t\tilde{\boldsymbol{E}}\boldsymbol{X} + \boldsymbol{t}^t\tilde{\boldsymbol{E}}\boldsymbol{X} = 0 \ . \tag{4.2}$$

This is the equation of a quadric passing through the origin.

Replacing $\tilde{\boldsymbol{E}}$, $\boldsymbol{R}$ and $\boldsymbol{t}$ by $\boldsymbol{E}$, $\tilde{\boldsymbol{R}}$ and $\tilde{\boldsymbol{t}}$ in (4.2) will give the equation of another quadric which is critical for the other motion.

Not only $\boldsymbol{X} = 0$ but also $\boldsymbol{Y} = 0$, that is $\boldsymbol{X} = -\boldsymbol{R}^t\boldsymbol{t}$ satisfies (4.2). These two points are the focal points of the two cameras in the relative orientation interpretation. All points satisfying $\tilde{\boldsymbol{E}}\boldsymbol{X} = \tilde{\boldsymbol{C}}\tilde{\boldsymbol{R}}\boldsymbol{X} = 0$ do also sit on the quadric. This is the equation for a straight line through the origin in the direction $\tilde{\boldsymbol{R}}^t\tilde{\boldsymbol{t}}$, meaning that the surface is a ruled quadric. The detailed studies of refs. [7] and [12] show that the ruled quadric can be a hyperboloid of one sheet, a saddle surface, a circular cylinder or two orthogonal planes but not an elliptic cylinder or a cone. The latter two can be critical for a differential, i.e. a very small motion, see [12].

To construct an example with points on this dangerous surface, generate $\boldsymbol{R}$, $\boldsymbol{t}$, $\tilde{\boldsymbol{R}}$, and $\tilde{\boldsymbol{t}}$, e.g. by random numbers. Choose repeatedly a perspective coordinate $\boldsymbol{x}$ and let λ be a scale factor so that $\boldsymbol{X} = \lambda\boldsymbol{x}$ and $\boldsymbol{Y} = \boldsymbol{R}\boldsymbol{X} + \boldsymbol{t} = \lambda\boldsymbol{R}\boldsymbol{x} + \boldsymbol{t}$. Inserting these in (4.2) gives the following linear equation for λ .

$$\lambda\boldsymbol{x}^t\boldsymbol{R}^t\tilde{\boldsymbol{E}}\boldsymbol{x} + \boldsymbol{t}^t\tilde{\boldsymbol{E}}\boldsymbol{x} = 0 \tag{4.3}$$

With this λ , $\boldsymbol{X} = \lambda\boldsymbol{x}$ sits on the dangerous surface. If $X_3 < 0$ or $Y_3 < 0$, discard the point and try again.

5. Two critical configurations

A point configuration can be critical either because first, the system of equations (2.5) is not solvable or later, because the system (3.1) is not solvable. Both are systems of linear equations so they become insolvable if they have linearly dependent equations. We shall give two theorems about point configurations that defeat the first system. The first theorem is about points on a straight line, so it is relevant for the ruled quadrics (4.2). The second theorem about points in a plane is only applicable when the ruled quadric degenerates to two planes but it is probably of greater practical relevance since man made objects often have a plane surface. We show in Appendix 1 that any plane is a dangerous surface by showing how to construct the matrix $\tilde{\boldsymbol{E}}$ so that all points in the plane satisfy (4.2).

Theorem 1. *Three separate points on a straight line give three linearly independent equations. Further points on the same line give no further linearly independent equations.*

Proof: Let the line be defined by $\boldsymbol{X}^1$ and $\boldsymbol{X}^2$ and let $\boldsymbol{X}$, be another points on the line. We have

$$\boldsymbol{X} = \lambda \boldsymbol{X}^1 + (1-\lambda)\boldsymbol{X}^2 \tag{5.1}$$

where every $\lambda \neq 0, 1$. By definition, $\boldsymbol{Y}^1 = \boldsymbol{R}\boldsymbol{X}^1 + \boldsymbol{t}$ and $\boldsymbol{Y}^2 = \boldsymbol{R}\boldsymbol{X}^2 + \boldsymbol{t}$. We get

$$\boldsymbol{Y} = \boldsymbol{R}\boldsymbol{X} + \boldsymbol{t} = \lambda \boldsymbol{Y}^1 + (1-\lambda)\boldsymbol{Y}^2 . \tag{5.2}$$

Insertion in equation (2.5) gives

$$\begin{aligned}(\boldsymbol{Y})^t\boldsymbol{E}\boldsymbol{X} &= (\lambda\boldsymbol{Y}^1 + (1-\lambda)\boldsymbol{Y}^2)\boldsymbol{E}(\lambda\boldsymbol{X}^1 + (1-\lambda)\boldsymbol{X}^2) = \\ &= \lambda^2(\boldsymbol{Y}^1)^t\boldsymbol{E}\boldsymbol{X}^1 + \lambda(1-\lambda)((\boldsymbol{Y}^1)^t\boldsymbol{E}\boldsymbol{X}^2 + (\boldsymbol{Y}^2)^t\boldsymbol{E}\boldsymbol{X}^1) + (1-\lambda)^2(\boldsymbol{Y}^2)^t\boldsymbol{E}\boldsymbol{X}^2 = 0\end{aligned} . \tag{5.3}$$

Since $(\boldsymbol{Y}^1)^t\boldsymbol{E}\boldsymbol{X}^1 = 0$ and $(\boldsymbol{Y}^2)^t\boldsymbol{E}\boldsymbol{X}^2 = 0$, equation (5.3) reduces to

$$(\boldsymbol{Y}^1)^t\boldsymbol{E}\boldsymbol{X}^2 + (\boldsymbol{Y}^2)^t\boldsymbol{E}\boldsymbol{X}^1 = 0 . \tag{5.4}$$

Since (5.4) is independent of λ , any other point on the line will also give the relation (5.4). This means that three or more points on the same straight line will give three and no more independent equations for $\boldsymbol{E}$. QED.

This theorem implies, for instance, that the 8-point algorithm will find the motion parameters from a configuration consisting of four points on a straight line and five generic points. The four points will give three, and the generic five equations (2.5). The 8-point algorithm, implemented as described in section 6, will find the eight independent equations and solve the problem. If one knows which points sit on the straight line, one can just discard one or more of them and run any of the algorithms.

Theorem 2. *Six or more points in the same plane give at most six linearly independent* *equations.*

Proof: The proof is analogous to the former proof. Let the plane be defined by $\boldsymbol{X}^1$, $\boldsymbol{X}^2$, and $\boldsymbol{X}^3$. A further point $\boldsymbol{X}$ in the plane can be written

$$\boldsymbol{X} = \lambda_1 \boldsymbol{X}^1 + \lambda_2 \boldsymbol{X}^2 + \lambda_3 \boldsymbol{X}^3 \qquad \text{where} \qquad \lambda_1 + \lambda_2 + \lambda_3 = 1 \; . \tag{5.5}$$

We will get $\boldsymbol{Y} = \lambda_1 \boldsymbol{Y}^1 + \lambda_2 \boldsymbol{Y}^2 + \lambda_3 \boldsymbol{Y}^3$ and

$$\begin{aligned} \boldsymbol{Y}^t \boldsymbol{E} \boldsymbol{X} = {} & \lambda_1^2 (\boldsymbol{Y}^1)^t \boldsymbol{E} \boldsymbol{X}^1 + \lambda_2^2 (\boldsymbol{Y}^2)^t \boldsymbol{E} \boldsymbol{X}^2 + \lambda_3^2 (\boldsymbol{Y}^3)^t \boldsymbol{E} \boldsymbol{X}^3 + \lambda_1 \lambda_2 ((\boldsymbol{Y}^1)^t \boldsymbol{E} \boldsymbol{X}^2 + (\boldsymbol{Y}^2)^t \boldsymbol{E} \boldsymbol{X}^2) + \\ & + \lambda_1 \lambda_3 ((\boldsymbol{Y}^1)^t \boldsymbol{E} \boldsymbol{X}^3 + (\boldsymbol{Y}^3)^t \boldsymbol{E} \boldsymbol{X}^1) + \lambda_2 \lambda_3 ((\boldsymbol{Y}^2)^t \boldsymbol{E} \boldsymbol{X}^3 + (\boldsymbol{Y}^3)^t \boldsymbol{E} \boldsymbol{X}^2) \end{aligned} \tag{5.6}$$

With three sets of different λ_i for points 4, 5, and 6, we get the following six linearly independent equations for $\boldsymbol{E}$.

$$\begin{aligned} (\boldsymbol{Y}^i)^t \boldsymbol{E} \boldsymbol{X}^i = 0 \quad & \text{for} \quad i = 1, 2, 3 \\ (\boldsymbol{Y}^i)^t \boldsymbol{E} \boldsymbol{X}^j + (\boldsymbol{Y}^j)^t \boldsymbol{E} \boldsymbol{X}^i = 0 \quad & \text{for} \quad i, j = 1, 2, 3 \; , \; i \neq j \end{aligned} \; . \tag{5.7}$$

Any further points in the plane will not give any further independent equations. QED.

Note 1: If no four of the points in a plane sit on a straight line, six or more points will give six independent equations.
Note 2: Four points known to sit in a plane are sufficient for determining the motion parameters, [4].
Note 3: Six points in a plane will defeat the six point algorithm even though the above theorem doesn't imply that.

6. Numerical handling of the coplanarity conditions,

We suggest using singular value decomposition (SVD), [2], for handling the coplanarity conditions (2.5) together with the normalization (2.6). The same procedure can be used from five to many points.
Using SVD implies working with orthogonal transformations of the linear system (2.5) and such transformations don't harm the numerical accuracy. With Gaussian elimination, which is common practice for linear systems, the accuracy can be very poor.
The purpose of the calculation is to determine the elements of the essential matrix $\boldsymbol{E}$ from the pairs of perspective coordinates $(\boldsymbol{x}^i, \boldsymbol{y}^i)$, $1 \leq i \leq n$.
Form a 9-vector

$$\boldsymbol{e}^t = (e_{11}, e_{12}, e_{13}, e_{21}, e_{22}, e_{23}, e_{31}, e_{32}, e_{33}) \tag{6.1}$$

with the elements to be determined. The normalization (2.6) can be written $||\boldsymbol{e}|| = 1$, where $|| \ ||$ is the ordinary l_2-norm.

Form an $n \times 9$ "design-matrix" $\boldsymbol{A}$ with row i equal to

$$y_1^i x_1^i,\ y_1^i x_2^i,\ y_1^i x_3^i,\ y_2^i x_1^i,\ y_2^i x_2^i,\ y_2^i x_3^i,\ y_3^i x_1^i,\ y_3^i x_2^i,\ y_3^i x_3^i \quad . \tag{6.2}$$

Then, the system (2.5) can be written

$$\boldsymbol{A}\boldsymbol{e} = \boldsymbol{0} \ . \tag{6.3}$$

The SVD of the $n \times 9$-matrix $\boldsymbol{A}$ is

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^t \ , \tag{6.4}$$

where $\boldsymbol{U}$ is an $n \times n$- and $\boldsymbol{V}$ is an 9×9-orthogonal matrix. S is an $n \times 9$-matrix having the nonnegative and nonincreasing singular values $s_1 \geq s_2 \geq s_3 \geq \ . \ . \ . \ \geq 0$ along the diagonal while the off-diagonal elements are equal to zero.

Consider the minimization problem

$$\min_{||\boldsymbol{e}|| = 1} ||\boldsymbol{A}\boldsymbol{e}|| \quad . \tag{6.5}$$

If there exists an $\boldsymbol{e}$ satisfying (6.3), the min in (6.5) will be zero and it is attained for this $\boldsymbol{e}$. Expanding (6.5), we get

$$\min_{||\boldsymbol{e}|| = 1} ||\boldsymbol{A}\boldsymbol{e}|| = \min_{||\boldsymbol{e}|| = 1} ||\boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^t\boldsymbol{e}|| = \min_{||\boldsymbol{e}|| = 1} ||\boldsymbol{S}\boldsymbol{V}^t\boldsymbol{e}|| = \min_{||\boldsymbol{f}|| = 1} ||\boldsymbol{S}\boldsymbol{f}|| \ , \tag{6.6}$$

where $\boldsymbol{f} = \boldsymbol{V}^t\boldsymbol{e}$.

First, consider the 8-point algorithm and assume that we have 8 point pairs ($n = 8$) and assume further that these data are non-critical so that there exist a unique solution. Then, $\boldsymbol{A}$ has rank = 8 so s_1 through s_8 are positive. The minimizer has the form

$$\boldsymbol{S}\boldsymbol{f} = (s_1 f_1, s_2 f_2, . \ , \ . \ , \ . \ , s_8 f_8)^t = \boldsymbol{0} \ . \tag{6.7}$$

meaning that $f_1 = f_2 = \ . \ . \ . \ = f_8 = 0$ and $f_9 = 1$. The minimizing $\boldsymbol{e}$ is

$$\boldsymbol{e} = \boldsymbol{V}\boldsymbol{f} = \text{the last column of } \boldsymbol{V} \ . \tag{6.8}$$

With 9 or more point pairs ($n \geq 9$), (6.5) gives the least squares solution to the system of equations (2.5), see [2]. The minimizer has the form

$$\boldsymbol{S}\boldsymbol{f} = (s_1 f_1, s_2 f_2, . \ , \ . \ , \ . \ , s_8 f_8, s_9 f_9)^t = \boldsymbol{0} \ . \tag{6.9}$$

The min of $||\boldsymbol{A}\boldsymbol{e}||$ equals the smallest singular value = s_9 and it is still attained for $f_9 = 1$. The minimizing $\boldsymbol{e}$ is given by (6.8).

With $n < 8$ point pairs, the minimum is zero and the minimizer is

$$\boldsymbol{Sf} = (s_1 f_1, s_2 f_2, \ldots, s_n f_n)^t = \mathbf{0} . \tag{6.10}$$

If the point pairs are non-critical s_1 through s_n are positive making $f_1 = f_2 = \ldots = f_n = 0$. We have $\|\boldsymbol{f}\| = 1$ but otherwise the values of f_{n+1} through f_9 are arbitrary meaning that any linear combination of columns $n+1$ through 9 of $\boldsymbol{V}$ minimizes $\|\boldsymbol{Ae}\|$. This implies that these f_j can be taken as the parameters λ_k in (3.2) and the numbers α_{ijk} of the same formula are sitting in the last $9-n$ columns of $\boldsymbol{V}$. If $n < 9$ and your computer code for SVD requires the matrix to have at least as many rows as columns, just pad the last rows with zeroes.

7. Numerical handling of condition (3.1).

For $n = 5$, 6, and 7, we have, respectively, 4, 3, and 2 parameters λ_k, and the coefficients α_{ijk} are used to construct the second linear system (3.1). This system has 9 rows, one for each combination of i and j. The number of columns is $\binom{11-n}{3}$, which equals 20, 10. and 4, respectively, for the three algorithms. The 20 unknowns of the of the 5-point algorithm can be combined in a special way which eventually produces all essential matrices, i.e. all solutions, to the problem, [9]. For $n = 6$, the 9 equations are just as many as are needed for determining the relative size of the 10 unknowns of the 6-point algorithm. The relative sizes of the three λ_k can be calculated from these unknowns. We shall describe this calculation in detail for the 7-point algorithm, which has only two λ_k and four unknowns.

Since the third power of $\boldsymbol{E}$ from (3.2) is inserted in (3.1), the unknowns of this system, which we denote u_i, will consist of cubic combinations of the parameters λ_1 and λ_2. There are four such combinations, namely, $u_1 = \lambda_1^3$, $u_2 = \lambda_1^2\lambda_2$, $u_3 = \lambda_1\lambda_2^2$, and $u_4 = \lambda_2^3$.

We use SVD as described in section 6 for handling this 9×4-system as well as the 9×10-system for the 6-point algorithm. The singular values of system (3.1) will be denoted σ_k. The advantages with SVD compared to using Gaussian elimination are

1) Numerical stability.

2) No need to select a subset of the equations for the calculation (in the 7-point case).

3) No need to normalize by e.g. putting $\lambda_1 = 1$, which doesn't work if $\lambda_1 = 0$.
4) Automatic detection of rank deficiency in the system.
Having obtained u_1 - u_4 , we have the following three estimates of λ_1/λ_2 : u_1/u_2 , u_2/u_3, and u_3/u_4 . We choose the estimate for which the sum of the squares of the numerator and the denominator is largest even though there may be better ways of doing this choice. If the first ratio is chosen, we put $\lambda_1 = u_1$ and $\lambda_2 = u_2$, and analogously for the other choices. In the 6-point algorithm, one gets six estmates of the double ratio $\lambda_1/\lambda_2/\lambda_3$ to choose from.

8. Detection of critical configurations

Using the SVD as described above, enables us to detect critical configurations like those described in section 5. To obtain a solution of the relative orientation problem, first the system of linear equations (2.5) must be solved. If $n < 8$ the obtained result is used to construct system (3.1), which in its turn must be solvable. Criticality originating from linear dependency between the object points will make the system of equations (2.5) unsolvable. Sets of object points sitting on a genuinely quadratic dangerous surface will pass system (2.5) but their criticality will make the system (3.1) unsolvable.
The singular values of the systems (2.5) and (3.1) are denoted s_i and σ_i, respectively. A linear system has a unique solution if and only if it has full rank. Since the rank of a matrix equals its number of positive singular values, we have the following
Theorem 3. *A point configuration is non-critical for*
a) the 6-point algorithm if and only if $s_8 > 0$ *and* $\sigma_9 > 0$.
b) the 7-point algorithm if and only if $s_8 > 0$ *and* $\sigma_3 > 0$.
c) the 8-point algorithm if and only if $s_8 > 0$.
A dangerous surface of type (4.2) can be of four types: (1) a hyperboloid of one sheet, (2) a hyperbolic paraboloid, (3) a circular cylinder or (4) two intersecting planes. One of the two planes of case (4) passes through the origin and will not be considered here. See [7] for a detailed study of the equations and other properties of the four cases. The quadratic forms describing the surfaces of type (2), (3), and (4) all have one eigenvalue equal to zero. For cases (3) and (4) to occur, further special conditions must be satisfied. These conditions are reflected in the occurence of singular values s_i and σ_k being equal to zero. We give in table 1 the $s_i = 0$ and $\sigma_k = 0$ that signal criticality for various combinations of points on and off a dangerous surface of type (1)-(4). Non-critical configurations are

marked OK.

Most of the facts presented in the table follow directly from the previous discussions. However, one would not expect $\sigma_5 = 0$ when the 6-point algorithm is applied to 6 points in a plane. Superficially, this case is not more degenerate than six points on a cylinder but it turns out that the linear dependence between the points introduces three unexpected relations in (3.1). Here, we will not digress on this fact.

The configurations in the table are assumed to be generic so that e.g. "all points on plane"-

Configuration	6-point algorithm	7-point algorithm	8-point algorithm
all points on surface of type (1) or (2)	$\sigma_9 = 0$	$\sigma_3 = 0$	$s_8 = 0$
all points but one on surface of type (1) or (2)	OK	OK	OK
all points on cylinder	$\sigma_8 = 0$	$\sigma_2 = 0$	$s_8 = 0$
all points but one on cylinder	OK	OK	OK
all points on plane	$\sigma_5 = 0$	$s_7 = 0$	$s_7 = 0$
all points but one on plane	OK	OK	$s_8 = 0$
all points but two on plane	OK	OK	OK

TABLE 1. Singular value signalling criticality for various number of points on and off a dangerous surface for the 6-, 7-, and 8-point algorithms. Non-critical configurations are marked OK. The results in the last column are valid for eight and more points. The points are assumed to be generic but for the condition given in the first column.

excludes the case that all the points actually sit on a straight line.

9. Finding all solutions for a critical configuration.

For six or more points, it has been shown, [5], [7], [12], that there may be up to three different solutions for a configuration having all its points sitting on a dangerous surface, whose equation has a quadratic term. For all points in a plane, there may be two solutions. These double or triple solutions can be calculated by the 5-point algorithm. The technique is to first feed five of the point pairs to the 5-point algorithm. It will produce up to ten essential matrices[1]. If all the points sit on a dangerous surface, eight of the essential matrices may correspond to point configurations in front of the two cameras and so constitute

a solution. (For generic data there will be at most six solutions.) Then, for each solution, one checks if the points not used by the 5-point algorithm satisfy that solution.
The only critical configurations that defeat the 5-point algorithm are those that have four or five points on the same straight line. They will make $s_5 = 0$ or $s_4 = 0$, respectively.

10. Noisy data.

When it comes to noisy data, no singular values will be exactly zero. For $n > 8$ points, which generally are needed with noisy data, s_9 depends on the size of the noise. More precisely, $s_9 / \sqrt{n-8}$ is an estimate of the residuals of the least squares process and can be used to estimate the standard deviation of the perspective coordinates, [8].
By sec. 6, $s_8 = 0$ for a critical configuration with exact data. For noisy data, a small value of s_8 indicates criticality but it is difficult to give a generally valid recommendation of what small means in this connection.
For a random set of noncritical problems which are similar in the sense that their noise is about the same and the solid angles of the objects are of the same magnitude, the s_8-values have a fairly narrow distribution. A problem having s_8 smaller than say one fifth of the median of this distribution can be suspected to be close to critical.
Any noncritical problem can be expected to have s_8 at least twice and usually more than ten times as big as s_9 . Since $s_8 \geq s_9$, the small s_8 of a critical problem will approach s_9 . A practical recommendation could be to consider a configuration as critical if $s_8 < 1.5 \cdot s_9$.
The indicators of low accuracy of the solution given in [8] are good at detecting nearly critical configurations.

11. Conclusions.

We have studied point configurations for which the relative orientation problem in photogrammetry or equivalenly the motion problem of computer vision is not uniquely solvable. Such configurations are called critical. The criticality of a point configuration depends on the algorithm used to solve the problem. The results about criticality are summarized in table 1. A numerically stable way of implementing the algorithms has been

1. The method descrbed in [9], which gives up to twelve essential matrices, can be improved by rearranging the equations.

suggested. So implemented, the algorithms are able to detect when a point configuration is critical.

12. Appendix 1. An arbitrary plane is dangerous.

We raise the question: Is any plane in front of the camera a dangerous surface for a given rotation $\boldsymbol{R}$ and translation $\boldsymbol{t}$? By Theorem 3, the answer is affirmative for seven or more points. The situation is more complicated for six points in a plane because then, system (2.5) is solvable while system (3.1) is not. We shall show that six points in a given plane is a critical configuration by describing how to construct an alternative motion $\tilde{\boldsymbol{R}}$ and $\tilde{\boldsymbol{t}}$ such that the corresponding dangerous surface (4.2) is two planes one of which is the given plane $\hat{\boldsymbol{N}}^t\boldsymbol{X} = d$. The case with a plane is studied in detail in [6]. We have borrowed some notations and ideas from [7], in which all kinds of dangerous surfaces are considered. We refer to the theory in [7] and just give the recipe for constructing the alternative motion:

1) Form $\hat{\boldsymbol{N}}_1 = \boldsymbol{R}\hat{\boldsymbol{N}}$ and $\boldsymbol{v} = \hat{\boldsymbol{N}}_1 \times \boldsymbol{t}$. Calculate the angle α between $\hat{\boldsymbol{N}}_1$ and $\boldsymbol{t}$.

2) Form $\phi = \tan^{-1}\left(\dfrac{\sin\alpha}{\cos\alpha + (2d)/\|\boldsymbol{t}\|}\right)$.

3) Form $\boldsymbol{N}_2 = \boldsymbol{v} \times \hat{\boldsymbol{N}}_1$ so that $\hat{\boldsymbol{N}}_1$ and $\hat{\boldsymbol{N}}_2$ is an orthonomal basis in the plane orthogonal to v.

4) Define $\tilde{\boldsymbol{t}} = \hat{\boldsymbol{N}}_1\cos\phi + \hat{\boldsymbol{N}}_2\sin\phi$.

5) Let $\boldsymbol{L}$ be the rotation matrix that rotates the angle 2ϕ around the axis $\boldsymbol{v}$.

6) Define $\tilde{\boldsymbol{R}} = \boldsymbol{L}\boldsymbol{R}$.

13. Appendix 2. An arbitrary cylinder is not dangerous.

We raise the question: Is every cylinder in front of the camera which passes the origin dangerous for a given rotation $\boldsymbol{R}$ and translation $\boldsymbol{t}$? Let the cylinder be described by its axis $\hat{\tau}$ and the point on the axis $\rho\hat{\boldsymbol{x}}_0$, where $\hat{\boldsymbol{x}}_0$ is orthogonal to $\hat{\tau}$ and ρ is the radius. The cylinder is in front of the camera if $\hat{\boldsymbol{x}}_0$ has a positive third component. The situation is as follows. There is a dangerous cylinder with the prescribed $\hat{\tau}$ and $\hat{\boldsymbol{x}}_0$ but ρ cannot be prescribed. We give the recipe for constructing the alternative motion $\tilde{R}$, $\tilde{t}$.

1) Put $\tilde{\boldsymbol{t}} = \boldsymbol{R}\hat{\tau}$.

2) Put $\boldsymbol{v} = \boldsymbol{t} \times \tilde{\boldsymbol{t}}$.

3) Determine $|\phi|$ from $\cos\phi = \hat{\boldsymbol{v}} \cdot \boldsymbol{R}\hat{\boldsymbol{x}}_0$. Let ϕ have the same sign as $\tilde{\boldsymbol{t}} \cdot \boldsymbol{R}\hat{\boldsymbol{x}}_0 \times \boldsymbol{v}$.

4) Let $\boldsymbol{L}$ be the rotation matrix that rotates the angle ϕ around the axis $\tilde{\boldsymbol{t}}$.

5) Put $\tilde{\boldsymbol{R}} = \boldsymbol{L}\boldsymbol{R}$.

By (4.2), the size of the cylinder is proportional to $\boldsymbol{t}$ but not to $\tilde{\boldsymbol{t}}$ so ρ cannot be chosen arbitrarily. We have $\rho = \|\boldsymbol{v}\| / (2|\sin\phi|)$.

14. References

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